

# TORIC VECTOR BUNDLES AND PARLIAMENTS OF POLYTOPES

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**ABSTRACT.** We introduce a collection of convex polytopes associated to a toric vector bundle on a smooth complete toric variety. We show that the lattice points in these polytopes correspond to generators for the space of global sections and we relate edges to jets. Using the polytopes, we also exhibit bundles that are ample but not globally generated, and bundles that are ample and globally generated but not very ample.

## 1. INTRODUCTION

The importance and prevalence of toric varieties stems from their calculability and their close relation to polyhedral objects. The challenge is to emulate this success and enlarge the class of varieties with both features. Rather than contemplating spherical varieties or all  $T$ -varieties, we extend the theory of toric varieties by studying torus-equivariant vector bundles and their projective bundles. Motivated by the ensuing simplifications in the toric dictionary between line bundles and polyhedra, we concentrate on bundles over a smooth complete toric variety. The goal of this paper is to give explicit polyhedral interpretations for properties of these bundles.

To accomplish this goal, we fix a smooth complete toric variety  $X$  determined by the fan  $\Sigma$ . Let  $M$  denote the character lattice of the dense torus in  $X$  and write  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  for the unique minimal generators of the rays in  $\Sigma$ . A **toric vector bundle** on  $X$  is a torus-equivariant locally-free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of finite rank. The celebrated Klyachko classification proves that  $\mathcal{E}$  corresponds to a finite-dimensional vector space  $E$  equipped with compatible decreasing filtrations  $E \supseteq \dots \supseteq E^{\mathbf{v}_i}(j) \supseteq E^{\mathbf{v}_i}(j+1) \supseteq \dots \supseteq 0$  where  $1 \leq i \leq n$  and  $j \in \mathbb{Z}$ ; see §2.2. This collection of linear subspaces embeds into the lattice of flats for a distinguished linear matroid  $\mathcal{M}_{\mathcal{E}}$ . For each vector  $\mathbf{e}$  in the ground set  $G(\mathcal{M}_{\mathcal{E}})$ , we introduce the convex polytope

$$P_{\mathbf{e}} := \{ \mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leq \max(j \in \mathbb{Z} : \mathbf{e} \in E^{\mathbf{v}_i}(j)) \text{ for all } 1 \leq i \leq n \}.$$

We call the set of  $P_{\mathbf{e}}$  the **parliament of polytopes** for  $\mathcal{E}$ ; for more detail see §2.3. Although the defining half-spaces for  $P_{\mathbf{e}}$  and the vectors  $\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})$  encode the filtrations, the polytopes themselves may be empty.

The following result gives the first substantive connection between the polytopes and the toric vector bundle.

**Proposition 1.1.** *The lattice points in the parliament of polytopes for  $\mathcal{E}$  correspond to the torus-equivariant generating set for the space of global sections of  $\mathcal{E}$ .*

Example 2.8 recovers the well-known polytope associated to a torus-equivariant line bundle on  $X$ . However, when the rank of  $\mathcal{E}$  is greater than 1, Example 2.10 demonstrates that the lattice points in the parliament of polytopes need not yield a basis for the space of global sections. This highlights the key difference between toric vector bundles of higher rank and line bundles—toric vector bundles depend on both the combinatorics of the polytopes  $P_{\mathbf{e}}$  and the properties of the

vectors  $\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})$ . For line bundles, we may overlook the vector indexing the polytope because linear algebra in a one-dimensional vector space is trivial. Our criterion for deciding whether a toric vector bundle is globally generated underscores this distinction.

To outline this result, consider a maximal cone  $\sigma \in \Sigma$ . The fibre of  $\mathcal{E}$  over the torus-fixed point  $x_{\sigma} \in X$  is generated by a basis  $\mathcal{B}_{\sigma} \subseteq G(\mathcal{M}_{\mathcal{E}})$  for  $E$ ; see §2.11. Since the restriction of  $\mathcal{E}$  to the affine open toric variety  $U_{\sigma}$  splits equivariantly as a direct sum of toric line bundles, these equivariant line bundles contribute a multiset  $\mathbf{u}(\sigma)$  of characters associated to  $\mathcal{E}$ .

**Theorem 1.2.** *A toric vector bundle is globally generated if and only if, for each maximal cone  $\sigma \in \Sigma$ , the associated characters  $\mathbf{u}(\sigma)$  correspond to vertices of the lattice polytopes in the parliament of polytopes indexed by the basis elements in  $\mathcal{B}_{\sigma}$ .*

Example 2.16 demonstrates that global generation is not simply a property of the individual polytopes in the parliament. As an ancillary observation, Example 3.6 shows that the higher-cohomology of globally-generated ample toric vector bundle may be nonzero.

The parliament of polytopes for  $\mathcal{E}$  gives new insights into the projective bundle  $\mathbb{P}(\mathcal{E})$ . This is particularly relevant for the positivity properties of  $\mathcal{E}$  defined by the corresponding attribute for the tautological line bundle  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ . For instance, we may picture the restriction of  $\mathcal{E}$  to a torus-invariant curve in  $X$  as the normalized distances between appropriately matched characters associated to  $\mathcal{E}$ ; for a more thorough explanation see §3.1. Hence, Theorem 2.1 in [HMP] allows us to quickly recognize ample and nef toric vector bundles. Exploiting our polyhedral interpretations, Example 3.4 exhibits a toric vector bundle  $\mathcal{F}$  on  $\mathbb{P}^2$  that is ample but not globally generated, and Example 4.7 showcases a toric vector bundle  $\mathcal{H}$  on  $\mathbb{P}^2$  that is ample and globally generated, but not very ample. Better still, Proposition 3.5 and Remark 4.10 prove that  $\mathcal{F}$  and  $\mathcal{H}$  have the minimal rank among all toric vector bundles on  $\mathbb{P}^d$  with the given traits. Beyond completely answering Question 7.5 in [HMP], these examples reinforce the generic prediction that versions of positivity that coincide for line bundles diverge for vector bundles of higher rank.

The discrete geometry within the parliament of polytopes nevertheless captures the positivity of jets and, capitalizing on this, we discover that several forms of higher-order positivity are equivalent for toric vector bundles. This conspicuously violates the generic prediction. To be more precise, a vector bundle  $\mathcal{E}$  separates  $k$ -jets for  $k \in \mathbb{N}$  if, for every closed point  $x \in X$  with maximal ideal  $\mathfrak{m}_x \subseteq \mathcal{O}_X$ , the natural map  $H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x^{k+1})$  is surjective; §4.1 develops and generalizes these ideas. Theorem 4.3, which we regard as a higher-order enhancement of Theorem 1.2, establishes that a toric vector bundle  $\mathcal{E}$  separates  $k$ -jets if and only if certain edges in the parliament of polytopes have length at least  $k$ . This leads to the following equivalences.

**Theorem 1.3.** *A toric vector bundle  $\mathcal{E}$  separates  $k$ -jets if and only if it is  $k$ -jet ample. Moreover,  $\mathcal{E}$  separates 1-jets if and only if it is very ample.*

In contrast with arbitrary vector bundles on a smooth projective variety, we see that these versions of positivity coincide for toric vector bundles. In addition, we recover the main theorems from [DiR] by specializing to line bundles.

**Future directions.** The introduction of the parliament of polytopes for a toric vector bundle suggests some enticing research projects. The most straightforward advances would provide polyhedral interpretations for other properties of toric vector bundles. For example, we suspect that a toric vector bundle is big if and only if some Minkowski sum of the polytopes in the parliament is full-dimensional. For a globally-generated toric vector bundle  $\mathcal{E}$ , the complete linear series of  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  maps the projective bundle  $\mathbb{P}(\mathcal{E})$  into projective space. Can one characterize the homogeneous equations of the image in terms of combinatorial commutative algebra? If so, then one expects a description of the initial ideals via regular triangulations; compare with §8 in [Stu]. Since there exists ample, but not globally generated, line bundles on varieties of the form  $\mathbb{P}(\mathcal{E})$ , this class of varieties makes an interesting testing ground for Fujita’s conjecture; see Conjecture 10.4.1 in [PAG2]. More ambitiously, for an ample toric vector bundle  $\mathcal{E}$ , one could even ask for an effective polyhedral bound on  $m \in \mathbb{N}$  such that  $\text{Sym}^m(\mathcal{E})$  is globally generated or very ample. Finally, we wonder if there are natural topological hypotheses on the parliament of polytopes which imply that all of the higher-cohomology groups vanish.

**Conventions.** Throughout the document,  $\mathbb{N}$  denotes the nonnegative integers and  $X$  is a smooth complete toric variety over the complex numbers  $\mathbb{C}$ . The linear subspace generated by the vectors  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$  in a  $\mathbb{C}$ -vector space is denoted by  $\text{Span}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m)$ , and the polyhedral cone generated by the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  in a  $\mathbb{R}$ -vector space is denoted by  $\text{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ .

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## 2. GLOBAL SECTIONS AND LATTICE POLYTOPES

In addition to collecting standard definitions and notation related to toric vector bundles, this section introduces explicit torus-equivariant generators for the global sections of the toric vector bundle that correspond to the lattice points in a collection of polytopes.

**2.1. The underlying smooth toric variety.** Let  $X$  be a smooth complete  $d$ -dimensional toric variety determined by the strongly convex rational polyhedral fan  $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$  where  $N$  is a lattice of rank  $d$ . The  $\mathbb{Z}$ -dual lattice is  $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  and  $T := \text{Spec } \mathbb{C}[M]$  is the dense algebraic torus acting on  $X$ . The  $j$ -dimensional cones of  $\Sigma$  form the set  $\Sigma(j)$ . For  $\sigma \in \Sigma(d)$ , the corresponding torus-fixed point is  $x_\sigma \in X$ . We order the 1-dimensional cones  $\Sigma(1)$  (also known as rays) and, for  $1 \leq i \leq n$ , we write  $\mathbf{v}_i \in N$  for the unique minimal generator of the  $i$ -th ray. The  $i$ -th ray corresponds to the prime torus-invariant divisor  $D_i$  on  $X$  and the divisors  $D_1, D_2, \dots, D_n$  generate the group  $\text{Div}_T(X) \cong \mathbb{Z}^n$  of torus-invariant divisors. Since  $X$  is complete, there is a short exact sequence

$$0 \longrightarrow M \xrightarrow{\text{div}} \text{Div}_T(X) \longrightarrow \text{Pic}(X) \longrightarrow 0$$

where  $\text{div } \mathbf{u} := \langle \mathbf{u}, \mathbf{v}_1 \rangle D_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle D_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle D_n$  and the second map is the projection from the group of divisors to the Picard group. The invertible sheaf or line bundle associated to a divisor  $D \in \text{Div}_T(X)$  is denoted by  $\mathcal{O}_X(D)$ . For more information on toric varieties, see [CLS] or [Ful].

**2.2. Torus-equivariant vector bundles.** A *toric vector bundle* is a locally-free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of finite rank equipped with a  $T$ -action that is compatible with the  $T$ -action on  $X$ . In other words, there exists a  $T$ -action on  $\mathbb{V}(\mathcal{E}) := \text{Spec}(\text{Sym } \mathcal{E})$  such that the projection  $\pi: \mathbb{V}(\mathcal{E}) \rightarrow X$  is  $T$ -equivariant and  $T$  acts linearly on the fibres. There is also an induced  $T$ -action on the  $\mathbb{C}$ -vector spaces of sections  $H^0(U_\sigma, \mathcal{E})$  for every  $\sigma \in \Sigma$ . For  $\mathbf{u} \in M$ , the trivial line bundle  $\mathcal{O}_X(\text{div } \mathbf{u})$  has a canonical equivariant structure. More precisely, for every  $\sigma \in \Sigma$ , we have  $H^0(U_\sigma, \mathcal{O}_X(\text{div } \mathbf{u})) := \mathbb{C}[\sigma^\vee \cap M] \cdot \chi^{-\mathbf{u}}$  where  $\chi^{\mathbf{u}}$  is the character of  $\mathbf{u} \in M$ . Thus, the torus  $T$  acts on  $\text{Span}(\chi^{\mathbf{u}}) \subseteq H^0(U_\sigma, \mathcal{O}_X)$  by  $\chi^{-\mathbf{u}}$ . As in [HMP], we follow the standard convention in invariant theory for the action of the group on the ring of functions; in the toric literature, the opposite sign convention is often employed. Every toric line bundle on  $U_\sigma$  is equivariantly isomorphic to some  $\mathcal{O}_X(\text{div } \mathbf{u})|_{U_\sigma}$  where the class  $\bar{\mathbf{u}}$  of  $\mathbf{u}$  in  $M_\sigma := M/(\sigma^\perp \cap M)$  is uniquely determined. Any toric vector bundle on an affine toric variety splits equivariantly as a sum of toric line bundles whose underlying line bundles are trivial; see Proposition 2.2 in [Pay1]. Hence, for each  $\sigma \in \Sigma$ , there is a unique multiset  $\mathbf{u}(\sigma) \subset M_\sigma$  such that  $\mathcal{E}|_{U_\sigma} \cong \bigoplus_{\bar{\mathbf{u}} \in \mathbf{u}(\sigma)} \mathcal{O}_X(\text{div } \mathbf{u})|_{U_\sigma}$  where  $\mathbf{u} \in M$  is any lift of  $\bar{\mathbf{u}}$ . If  $\sigma$  is a  $d$ -dimensional cone, then the multiset  $\mathbf{u}(\sigma)$  is uniquely determined by  $\mathcal{E}$  and  $\sigma$ .

Toric vector bundles are classified by Theorem 0.1.1 in [Kly] via canonical filtrations. To summarize this classification, let  $E$  be the fibre of  $\mathcal{E}$  over the identity of the torus  $T$ . The category of toric vector bundles on  $X$  is naturally equivalent to the category of finite-dimensional  $\mathbb{C}$ -vector spaces  $E$  with decreasing filtrations  $\{E^{\mathbf{v}_i}(j)\}_{j \in \mathbb{Z}}$ , where  $1 \leq i \leq n$ , that satisfy the compatibility condition:

$$(2.2.1) \quad \begin{aligned} &\text{For each } \sigma \in \Sigma(d), \text{ there exists a decomposition } E = \bigoplus_{\mathbf{u} \in \mathbf{u}(\sigma)} E_{\mathbf{u}} \\ &\text{such that } E^{\mathbf{v}_i}(j) = \sum_{\langle \mathbf{u}, \mathbf{v}_i \rangle \geq j} E_{\mathbf{u}}. \end{aligned}$$

For a self-contained exposition of this classification, see §2.3 in [Pay1]. The decreasing filtrations associated to a toric vector bundle  $\mathcal{E}$  are defined as follows. For every  $\sigma \in \Sigma$  and every  $\mathbf{u} \in M$ , evaluating sections at the identity gives an injective map  $H^0(U_\sigma, \mathcal{E})_{\mathbf{u}} \hookrightarrow E$ . The image of this map is  $E_{\mathbf{u}}^\sigma \subseteq E$ . Following §4.2 in [Pay2], we define a linear subspace  $E^{\mathbf{v}}(j) \subseteq E$  for every  $\mathbf{v} \in N$  and every  $j \in \mathbb{Z}$ . Since  $X$  is complete, there exists a unique cone  $\sigma \in \Sigma$  containing  $\mathbf{v}$  in its relative interior. Set  $E^{\mathbf{v}}(j) := \sum_{\langle \mathbf{u}, \mathbf{v} \rangle \geq j} E_{\mathbf{u}}^\sigma$ . For a fixed  $\mathbf{v} \in N$ , the family of linear subspaces  $\{E^{\mathbf{v}}(j)\}_{j \in \mathbb{Z}}$  give a decreasing filtration of  $E$ .

The filtrations associated to  $\mathcal{E}$  have a second interpretation. For a cone  $\sigma \in \Sigma$ , suppose that we have  $\mathcal{E}|_{U_\sigma} \cong \bigoplus_{\bar{\mathbf{u}} \in \mathbf{u}(\sigma)} \mathcal{O}_X(\text{div } \mathbf{u})|_{U_\sigma}$ . If  $L_{\mathbf{u}} \subseteq E$  is the fibre of  $\mathcal{O}_X(\text{div } \mathbf{u})$  over the identity of  $T$ , then we have a decomposition  $E = \bigoplus_{\bar{\mathbf{u}} \in \mathbf{u}(\sigma)} L_{\mathbf{u}}$ . Hence, the linear subspace  $E_{\mathbf{u}}^\sigma$  is spanned by the  $L_{\mathbf{u}}$  for which  $\mathbf{u} - \mathbf{u}' \in \sigma^\vee$  and  $E^{\mathbf{v}}(j) = \bigoplus_{\langle \mathbf{u}, \mathbf{v} \rangle \geq j} L_{\mathbf{u}}$ . Moreover, for every  $d$ -dimensional cone  $\sigma \in \Sigma$ , there exists a subset  $\tilde{\mathbf{u}}(\sigma) \subset M$  and a decomposition  $E = \bigoplus_{\mathbf{u} \in \tilde{\mathbf{u}}(\sigma)} E_{\mathbf{u}}$  such that, for every  $\mathbf{v} \in \sigma$  and every  $j \in \mathbb{Z}$ , we have  $E^{\mathbf{v}}(j) = \bigoplus_{\langle \mathbf{u}, \mathbf{v} \rangle \geq j} E_{\mathbf{u}}$ . It follows that  $E_{\mathbf{u}} = \bigoplus_{\bar{\mathbf{u}} \in \mathbf{u}(\sigma)} L_{\mathbf{u}}$ , so  $\dim E_{\mathbf{u}}$  equals the multiplicity of  $\mathbf{u}$  in the multiset  $\mathbf{u}(\sigma)$  and  $\tilde{\mathbf{u}}(\sigma)$  is the set of elements appearing in  $\mathbf{u}(\sigma)$ .

**2.3. Associated convex polytopes.** Each equivariant line bundle  $\mathcal{L}$  on  $X$  corresponds to a rational convex polytope. We generalize this notion by attaching a collection of convex polytopes to a toric vector bundle  $\mathcal{E}$ . These polytopes are indexed by the vectors in the ground set of a linear matroid.

To describe this matroid, observe that  $\mathcal{E}$  determines the finite poset  $L(\mathcal{E})$  consisting of the linear subspaces  $V := \bigcap_{i=1}^n E^{\mathbf{v}_i}(j_i) \subseteq E$ , where  $j_i \in \mathbb{Z}$ , ordered by inclusion. Since we have the decreasing filtration  $E \supseteq \cdots \supseteq E^{\mathbf{v}_i}(j) \supseteq E^{\mathbf{v}_i}(j+1) \supseteq \cdots \supseteq 0$ , for each  $1 \leq i \leq n$ , it follows that  $0, E \in L(\mathcal{E})$  and the poset  $L(\mathcal{E})$  is closed under intersection. Hence, any subset  $V_1, V_2, \dots, V_k$  of linear subspaces in  $L(\mathcal{E})$  has a unique meet  $V_1 \cap V_2 \cap \cdots \cap V_k$  and a unique join  $\bigcap_{V_1+V_2+\dots+V_k \subseteq W} W$ , so  $L(\mathcal{E})$  is a complete lattice. The ensuing proposition shows that  $L(\mathcal{E})$  is embedded into the lattice  $L(\mathcal{M}_{\mathcal{E}})$  of flats for a distinguished linear matroid  $\mathcal{M}_{\mathcal{E}}$ .

**Proposition 2.4.** *There exists a unique matroid  $\mathcal{M}_{\mathcal{E}}$  such that*

- (a)  *$L(\mathcal{E})$  is isomorphic to a sublattice of  $L(\mathcal{M}_{\mathcal{E}})$  where  $V \in L(\mathcal{E})$  corresponds to  $R_V \in L(\mathcal{M}_{\mathcal{E}})$ , and*
- (b) *for every  $S \in L(\mathcal{M}_{\mathcal{E}})$ , we have  $\text{rank}(S) = \min_{R_V \subseteq S} (\dim(V) + |S \setminus R_V|)$ .*

*Moreover, the matroid  $\mathcal{M}_{\mathcal{E}}$  is representable over  $\mathbb{C}$ .*

In other words, the matroid  $\mathcal{M}_{\mathcal{E}}$  is the free expansion of  $L(\mathcal{E})$ ; see §10.2 in [Whi]. Using the terminology from linear subspace arrangements (and ordering subspaces by reversed inclusion), Proposition 2.4 is also equivalent to Theorem 4.9 in [Zie].

*Proof.* We inductively construct a linear matroid with the required embedding. For  $\ell \in \mathbb{N}$  satisfying  $0 \leq \ell \leq r$ , let  $W$  be the linear subspace in  $E$  generated by all  $W' \in L(\mathcal{E})$  with  $\dim(W') < \ell$ . For each  $V \in L(\mathcal{E})$  with  $\dim(V) = \ell$ , choose a basis for a complementary subspace to  $V \cap W$  in  $V$ . Let  $\mathcal{M}_{\mathcal{E}}$  be the linear matroid whose ground set  $G(\mathcal{M}_{\mathcal{E}})$  is the union of these bases; see §1.1.B in [Whi]. By construction, each linear subspace  $V$  in  $L(\mathcal{E})$  is generated by a subset of  $G(\mathcal{M}_{\mathcal{E}})$ , and the maximal subset with this property is the flat  $R_V$  in  $\mathcal{M}_{\mathcal{E}}$  corresponding to  $V$ . It follows that  $\text{rank}(R_V) = \dim(V)$  and the induced map from  $L(\mathcal{E})$  into  $L(\mathcal{M}_{\mathcal{E}})$  is an embedding of lattices. For any linear matroid, any inclusion  $R \subseteq S$  of flats implies that  $\text{rank}(S) \leq \text{rank}(R) + |S \setminus R|$ . If  $R_V$  is the largest flat in  $L(\mathcal{M}_{\mathcal{E}})$  that is both contained in a given flat  $S \in L(\mathcal{M}_{\mathcal{E}})$  and corresponds to linear subspace  $V \in L(\mathcal{E})$ , then we have  $\text{Span}(S) = V \oplus \text{Span}(S \setminus R_V)$ , so  $\text{rank}(S) = \dim(V) + |S \setminus R_V|$ . Since a matroid is characterized by its rank function, any matroid satisfying the two conditions must be isomorphic to the constructed matroid.  $\square$

**Remark 2.5.** The proof of Proposition 2.4 yields an algorithm for choosing vectors  $G(\mathcal{M}_{\mathcal{E}})$  that represent the intrinsic matroid  $\mathcal{M}_{\mathcal{E}}$ . Nevertheless, the relation  $\mathbf{e} \in V$  for  $\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})$  and  $V \in L(\mathcal{E})$  depends only on the matroid  $\mathcal{M}_{\mathcal{E}}$  and not on the choice of a representation.

**Remark 2.6.** For each  $\sigma \in \Sigma(d)$ , the compatibility condition (2.2.1) is equivalent to saying that the sublattice of  $L(\mathcal{E})$  consisting of the linear subspaces  $\bigcap_{\mathbf{v}_i \in \sigma} E^{\mathbf{v}_i}(j_i)$ , where  $j_i \in \mathbb{Z}$ , is distributive; see Remark 2.2.2 in [Kly]. Hence, the ground set  $G(\mathcal{M}_{\mathcal{E}})$  contains a distinguished basis  $\mathcal{B}_{\sigma}$  of  $E$  such that the component  $E^{\mathbf{v}_i}(j)$ , where  $\mathbf{v}_i \in \sigma$  and  $j \in \mathbb{Z}$ , is a direct sum of coordinate subspaces.

For each vector  $\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})$ , the associated convex polytope is

$$P_{\mathbf{e}} := \{ \mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leq \max(j \in \mathbb{Z} : \mathbf{e} \in E^{\mathbf{v}_i}(j)) \text{ for all } 1 \leq i \leq n \}.$$



Using the collective noun for owls, we call the collection  $\{P_{\mathbf{e}} : \mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})\}$  the *parliament of polytopes* for  $\mathcal{E}$ . The number of polytopes in the parliament for  $\mathcal{E}$  is at least the rank of  $\mathcal{E}$ .

**Remark 2.7.** A toric vector bundle  $\mathcal{E}$  splits equivariantly into a direct sum of line bundles if and only if there is a basis  $\mathcal{B}$  of  $E$  such that  $\mathcal{B} = \mathcal{B}_{\sigma}$  for all  $\sigma \in \Sigma(d)$ . Rephrasing this, we see that the number of polytopes in the parliament for  $\mathcal{E}$  equals the rank if and only if  $\mathcal{E}$  splits equivariantly into a direct sum of line bundles.

Extending the renowned theorem for line bundles on  $X$ , we have the following interpretation for the lattice points in a parliament of polytopes.

**Proposition 1.1.** *The lattice points in the parliament of polytopes for  $\mathcal{E}$  correspond to the torus-equivariant generating set for the space of global sections of  $\mathcal{E}$ ;*

$$H^0(X, \mathcal{E}) \cong \sum_{\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})} \text{Span}(\mathbf{e} \otimes \chi^{-\mathbf{u}} : \mathbf{u} \in P_{\mathbf{e}} \cap M).$$

*Proof of Proposition 1.1.* The torus-action on the space of global sections yields a decomposition into  $\chi^{\mathbf{u}}$ -isotypical components  $H^0(X, \mathcal{E})_{\mathbf{u}}$  for  $\mathbf{u} \in M$ . Moreover, the torus acts on  $H^0(X, \mathcal{E})_{\mathbf{u}}$  via the character  $\chi^{\mathbf{u}}$  and we have  $H^0(X, \mathcal{E}) = \bigoplus_{\mathbf{u} \in M} H^0(X, \mathcal{E})_{\mathbf{u}}$ . Since  $X$  is complete, at most finitely many of the isotypical components are nonzero. Following Corollary 4.1.3 in [Kly], evaluation at the identity of the torus gives a canonical isomorphism

$$H^0(X, \mathcal{E})_{\mathbf{u}} = \bigcap_{\sigma \in \Sigma(d)} H^0(U_{\sigma}, \mathcal{E})_{\mathbf{u}} \xrightarrow{\cong} \bigcap_{\sigma \in \Sigma(d)} E_{\mathbf{u}}^{\sigma} = \bigcap_{\mathbf{v} \in N} E^{\mathbf{v}}(\langle \mathbf{u}, \mathbf{v} \rangle) = \bigcap_{i=1}^n E^{\mathbf{v}_i}(\langle \mathbf{u}, \mathbf{v}_i \rangle).$$

Since the linear subspace  $V_{\mathbf{u}} := \bigcap_{i=1}^n E^{\mathbf{v}_i}(\langle \mathbf{u}, \mathbf{v}_i \rangle)$  belongs to the lattice  $L(\mathcal{E})$ , Proposition 2.4 shows that there exists a flat  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m\}$  in the matroid  $\mathcal{M}_{\mathcal{E}}$  such that  $\text{Span}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m) = V_{\mathbf{u}}$ . Hence, we obtain  $H^0(X, \mathcal{E})_{\mathbf{u}} \cong \sum_{\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})} \text{Span}(\mathbf{e} \otimes \chi^{-\mathbf{u}} : \mathbf{e} \in V_{\mathbf{u}})$ . Because we have

$$\begin{aligned} \mathbf{e} \in V_{\mathbf{u}} &\iff \mathbf{e} \in E^{\mathbf{v}_i}(\langle \mathbf{u}, \mathbf{v}_i \rangle) && \text{for all } 1 \leq i \leq n \\ &\iff \langle \mathbf{u}, \mathbf{v}_i \rangle \leq \max(j \in \mathbb{Z} : \mathbf{e} \in E^{\mathbf{v}_i}(j)) && \text{for all } 1 \leq i \leq n \\ &\iff \mathbf{u} \in P_{\mathbf{e}} \cap M, \end{aligned}$$

we conclude that  $H^0(X, \mathcal{E})_{\mathbf{u}} \cong \sum_{\mathbf{e} \in G(\mathcal{M}_{\mathcal{E}})} \text{Span}(\mathbf{e} \otimes \chi^{-\mathbf{u}} : \mathbf{u} \in P_{\mathbf{e}} \cap M)$ .  $\square$

As expected, we recover the well-known description for the global sections of a line bundle.

**Example 2.8.** Every line bundle  $\mathcal{L}$  on a smooth toric variety  $X$  equals  $\mathcal{O}_X(D)$  for some torus-invariant divisor  $D = a_1 D_1 + a_2 D_2 + \dots + a_n D_n$ . Theorem 6.1.7 in [CLS] establishes that the Cartier divisor  $D$  is determined by a collection  $\{\mathbf{u}_{\sigma} \in M : \sigma \in \Sigma(d)\}$ , so we obtain  $\mathbf{u}(\sigma) = \{\mathbf{u}_{\sigma}\}$  for all  $\sigma \in \Sigma(d)$ . The associated continuous piecewise linear function  $\varphi_D : N_{\mathbb{R}} \rightarrow \mathbb{R}$  satisfies  $\varphi_D(\mathbf{v}_i) = -a_i$  and  $\varphi_D(\mathbf{v}) = \langle \mathbf{u}_{\sigma}, \mathbf{v} \rangle$  for all  $\mathbf{v} \in \sigma$ . Following §2.3.1 in [Kly], the decreasing filtrations corresponding to  $\mathcal{L}$  are

$$E^{\mathbf{v}_i}(j) := \begin{cases} \mathbb{C} & \text{if } j \leq a_i \\ 0 & \text{if } j > a_i \end{cases} \quad \text{for all } 1 \leq i \leq n.$$

If  $\mathbf{e}$  is any nonzero vector in  $E = \mathbb{C}$ , then we have  $G(\mathcal{M}_{\mathcal{E}}) = \{\mathbf{e}\}$  and  $P_{\mathbf{e}} = \{\mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leq a_i\}$ . It follows that  $E_{\mathbf{u}_{\sigma}} = E = \mathbb{C}$  for all  $\sigma \in \Sigma(d)$ , so  $H^0(X, \mathcal{L})_{\mathbf{u}} = \mathbb{C}$  when  $\langle \mathbf{u}, \mathbf{v}_i \rangle \leq a_i$  for all  $1 \leq i \leq n$  and  $H^0(X, \mathcal{L})_{\mathbf{u}} = 0$  otherwise. Therefore, we have  $H^0(X, \mathcal{L}) = \bigoplus_{\mathbf{u} \in P_{\mathbf{e}} \cap M} \text{Span}(\mathbf{e} \otimes \chi^{-\mathbf{u}})$ . Notice that we use the opposite sign convention when compared to either §6.1 in [CLS] or §3.4 in [Ful].  $\diamond$

Our second example shows that the ground set  $G(\mathcal{M}_{\mathcal{E}})$  may be strictly larger than the union  $\bigcup_{\sigma \in \Sigma(d)} \mathcal{B}_{\sigma}$  of the bases for  $E$  which split the filtrations over the maximal cones.

**Example 2.9.** To describe a toric vector bundle  $\mathcal{E}$  of rank 3 on  $\mathbb{P}^1 \times \mathbb{P}^1$ , we first specify the underlying fan. The primitive lattice points on the rays are  $\mathbf{v}_1 = (1, 0)$ ,  $\mathbf{v}_2 = (0, 1)$ ,  $\mathbf{v}_3 = (-1, 0)$ ,  $\mathbf{v}_4 = (0, -1)$  and the maximal cones  $\sigma_{1,2} = \text{pos}(\mathbf{v}_1, \mathbf{v}_2)$ ,  $\sigma_{2,3} = \text{pos}(\mathbf{v}_2, \mathbf{v}_3)$ ,  $\sigma_{3,4} = \text{pos}(\mathbf{v}_3, \mathbf{v}_4)$ ,  $\sigma_{1,4} = \text{pos}(\mathbf{v}_1, \mathbf{v}_4)$ . If  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  denotes the standard basis of  $E = \mathbb{C}^3$ , then the decreasing filtrations defining  $\mathcal{E}$  are

$$\begin{aligned} E^{\mathbf{v}_1}(j) &= \begin{cases} E & \text{if } j \leq -1 \\ \text{Span}(\mathbf{e}_1, \mathbf{e}_2) & \text{if } -1 < j \leq 0 \\ \text{Span}(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } 0 < j \leq 1 \\ 0 & \text{if } 1 < j \end{cases} & E^{\mathbf{v}_3}(j) &= \begin{cases} E & \text{if } j \leq -1 \\ \text{Span}(\mathbf{e}_1, \mathbf{e}_3) & \text{if } -1 < j \leq 0 \\ \text{Span}(\mathbf{e}_1 + \mathbf{e}_3) & \text{if } 0 < j \leq 1 \\ 0 & \text{if } 1 < j \end{cases} \\ E^{\mathbf{v}_2}(j) &= \begin{cases} E & \text{if } j \leq 0 \\ \text{Span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if } 0 < j \leq 1 \\ \text{Span}(\mathbf{e}_2) & \text{if } 1 < j \leq 2 \\ 0 & \text{if } 2 < j \end{cases} & E^{\mathbf{v}_4}(j) &= \begin{cases} E & \text{if } j \leq 0 \\ \text{Span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if } 0 < j \leq 1 \\ \text{Span}(\mathbf{e}_2) & \text{if } 1 < j \leq 2 \\ 0 & \text{if } 2 < j \end{cases} \end{aligned}$$

It follows that  $G(\mathcal{M}_{\mathcal{E}}) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3\}$ , the associated characters and bases are

$$\begin{aligned} \mathbf{u}(\sigma_{1,2}) &= \{(1, 0), (0, 2), (-1, 1)\} & \mathcal{B}_{\sigma_{1,2}} &= \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_3\} \\ \mathbf{u}(\sigma_{2,3}) &= \{(-1, 0), (1, 2), (0, 1)\} & \mathcal{B}_{\sigma_{2,3}} &= \{\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3\} \\ \mathbf{u}(\sigma_{3,4}) &= \{(-1, 0), (1, -2), (0, -1)\} & \mathcal{B}_{\sigma_{3,4}} &= \{\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3\} \\ \mathbf{u}(\sigma_{1,4}) &= \{(0, -2), (1, 0), (-1, -1)\} & \mathcal{B}_{\sigma_{1,4}} &= \{\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3\}, \end{aligned}$$

and the convex polytopes are  $P_{\mathbf{e}_1} = \text{Conv}((0, 0))$ ,  $P_{\mathbf{e}_2} = \emptyset$ ,  $P_{\mathbf{e}_3} = \emptyset$ ,  $P_{\mathbf{e}_1 + \mathbf{e}_2} = \text{Conv}((1, 0))$  and  $P_{\mathbf{e}_1 + \mathbf{e}_3} = \text{Conv}((-1, 0))$ . Although  $\mathbf{e}_1 \notin \bigcup_{\sigma \in \Sigma(2)} \mathcal{B}_{\sigma}$ , we have  $\text{Span}(\mathbf{e}_1) = E^{\mathbf{v}_1}(0) \cap E^{\mathbf{v}_3}(0)$ ; there is no choice of basis for  $E$  which splits the filtrations along  $\mathbf{v}_1$  or  $\mathbf{v}_3$  and contains  $\mathbf{e}_1$ .  $\diamond$

The lattice points in the parliament of polytopes for a toric vector bundle correspond to a basis if and only if, for all  $\mathbf{u} \in M$ , the subset  $\{\mathbf{e} \in E : \mathbf{u} \in P_{\mathbf{e}}\}$  is linearly independent. The subsequent example illustrates how a single lattice point can correspond to a linearly dependent collection of global sections.

**Example 2.10.** Consider the tangent bundle  $\mathcal{T}_{\mathbb{P}^d}$  on  $\mathbb{P}^d$ . To be more explicit, we describe the corresponding fan: the primitive lattice vector  $\mathbf{v}_i$  generating the  $i$ -th ray equals the  $i$ -th standard basis vector in  $\mathbb{C}^d$  for  $1 \leq i \leq d$ , the unique additional ray is generated by  $\mathbf{v}_{d+1} := -\mathbf{v}_1 - \cdots - \mathbf{v}_d$ , and the maximal cones are  $\sigma_i := \text{pos}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{d+1})$  for  $1 \leq i \leq d+1$ ; compare with Example 3.1.10 in [CLS] or §1.4 in [Ful]. If we identify the fibre  $E$  of  $\mathcal{T}_{\mathbb{P}^d}$  over the identity of the

torus with  $N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^d$  as done in §2.3.5 of [Kly], then the decreasing filtrations defining  $\mathcal{T}_{\mathbb{P}^d}$  are

$$E^{\mathbf{v}_i}(j) = \begin{cases} E & \text{if } j \leq 0 \\ \text{Span}(\mathbf{v}_i) & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases} \quad \text{for } 1 \leq i \leq d+1.$$

Writing  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$  for the dual basis of  $M$  associated to the basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \in N$ , we have  $\mathbf{u}(\sigma_i) = \{\mathbf{w}_1 - \mathbf{w}_i, \mathbf{w}_2 - \mathbf{w}_i, \dots, \mathbf{w}_{i-1} - \mathbf{w}_i, -\mathbf{w}_i, \mathbf{w}_{i+1} - \mathbf{w}_i, \mathbf{w}_{i+2} - \mathbf{w}_i, \dots, \mathbf{w}_d - \mathbf{w}_i\}$  for  $1 \leq i \leq d$ , and  $\mathbf{u}(\sigma_{d+1}) = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}$ . Hence, we have  $G(\mathcal{M}_{\mathcal{T}_{\mathbb{P}^d}}) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$  and the convex polytopes are  $P_{\mathbf{v}_i} = \{\mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leq 1 \text{ and } \langle \mathbf{u}, \mathbf{v}_j \rangle \leq 0 \text{ for all } j \neq i\}$ . The lattice points in the parliament of polytopes for  $\mathcal{T}_{\mathbb{P}^d}$  correspond to the following  $(d+1)^2$  global sections:  $\mathbf{v}_i \otimes \chi^{\mathbf{w}_j - \mathbf{w}_i}$  for  $1 \leq i, j \leq d$ ,  $\mathbf{v}_i \otimes \chi^{-\mathbf{w}_i}$  for  $1 \leq i \leq d$ ,  $\mathbf{v}_{d+1} \otimes \chi^{\mathbf{w}_i}$  for  $1 \leq i \leq d$ , and  $\mathbf{v}_{d+1} \otimes \chi^{\mathbf{0}}$ . The origin  $\mathbf{0} \in M$  is therefore contained in  $d+1$  polytopes, which yields  $d+1$  global sections in a  $d$ -dimensional vector space.

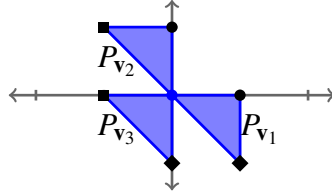


FIGURE 2.10.1. The parliament of polytopes for  $\mathcal{T}_{\mathbb{P}^2}$

When  $d = 2$ , it is possible to visualize the parliament of polytopes. In this case, the associated characters are  $\mathbf{u}(\sigma_1) = \{(-1, 0), (-1, 1)\}$ ,  $\mathbf{u}(\sigma_2) = \{(1, -1), (0, -1)\}$ ,  $\mathbf{u}(\sigma_3) = \{(1, 0), (0, 1)\}$ , and the convex polytopes are  $P_{\mathbf{v}_1} = \text{Conv}((0, 0), (1, 0), (1, -1))$ ,  $P_{\mathbf{v}_2} = \text{Conv}((0, 0), (0, 1), (-1, 1))$ ,  $P_{\mathbf{v}_3} = \text{Conv}((0, 0), (-1, 0), (0, -1))$ . In Figure 2.10.1, the associated characters are represented by squares, diamonds, and black circles respectively. The polytopes are represented by blue triangles and the other lattice point lying in the polytopes is represented by a blue circle.  $\diamond$

**2.11. Evaluating Sections.** To detect the global generation of a toric vector bundle  $\mathcal{E}$  in its parliament of polytopes, we need a local description of a global section in coordinates near a torus-fixed point. Fix a maximal cone  $\sigma \in \Sigma(d)$ . By reordering the rays (if necessary), we may assume that  $\sigma = \text{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$ . Since  $X$  is smooth, the minimal generators  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$  of the dual cone  $\sigma^\vee$  form a  $\mathbb{Z}$ -basis for  $M$ . By indexing the multiset of associated characters, we have  $\mathbf{u}(\sigma) = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ . Following §6.3 in [Kly], we identify the fibre of  $\mathcal{E}$  over the torus-fixed point  $x_\sigma \in X$  with the vector space

$$(2.11.2) \quad \mathcal{E}_{x_\sigma} \cong \bigoplus_{\mathbf{u} \in \mathbf{u}(\sigma)} \frac{E_{\mathbf{u}}^\sigma}{E_{>\mathbf{u}}^\sigma},$$

where  $E_{\mathbf{u}}^\sigma = \bigcap_{i=1}^d E^{\mathbf{v}_i}(\langle \mathbf{u}, \mathbf{v}_i \rangle)$  and  $E_{>\mathbf{u}}^\sigma := \sum_{\mathbf{0} \neq \mathbf{u}' - \mathbf{u} \in \sigma^\vee} E_{\mathbf{u}'}^\sigma$ . Remark 2.6 shows that  $E_{\mathbf{u}}^\sigma$  and  $E_{>\mathbf{u}}^\sigma$  correspond to flats in the matroid  $\mathcal{M}_{\mathcal{E}}$  that are contained in the basis  $\mathcal{B}_\sigma$ . As a consequence, this identification determines an injective map sending  $\mathbf{u}_\ell \in \mathbf{u}(\sigma)$  to  $\mathbf{e}_{\ell, \sigma} \in G(\mathcal{M}_{\mathcal{E}})$  such that



$\mathcal{B}_\sigma = \{\mathbf{e}_{1,\sigma}, \mathbf{e}_{2,\sigma}, \dots, \mathbf{e}_{r,\sigma}\}$ ; equivalently, there is a bijection between the multiset  $\mathbf{u}(\sigma)$  and the basis  $\mathcal{B}_\sigma$ . More precisely, if the multiset  $\mathbf{u}(\sigma)$  equals its underlying set  $\tilde{\mathbf{u}}(\sigma)$ , then the injective map is uniquely determined. However, if some character in the multiset  $\mathbf{u}(\sigma)$  has multiplicity greater than one, then we choose an injective map that is compatible with the identification. Despite our notation, the vector  $\mathbf{e}_{\ell,\sigma} \in E$  is not simply a function of the integer  $\ell \in \{1, 2, \dots, r\}$  and the cone  $\sigma \in \Sigma(d)$ , but depends on our indexing of the multiset  $\mathbf{u}(\sigma)$  and our choice of injective map.

The quotient space  $E_{\mathbf{u}}^\sigma / E_{>\mathbf{u}}^\sigma$  appearing in (2.11.2) has useful polyhedral implications.

**Lemma 2.12.** *Let  $\sigma = \text{pos}(\mathbf{v}_1, \dots, \mathbf{v}_d) \in \Sigma(d)$  be a maximal cone and let  $\mathbf{u}(\sigma) = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$  be the multiset of associated characters. If  $\varphi_{\ell,\sigma}(\mathbf{v}) := \max(\langle \mathbf{u}, \mathbf{v} \rangle : \mathbf{u} \in P_{\mathbf{e}_{\ell,\sigma}})$  is the piecewise linear support function corresponding to the convex polytope  $P_{\mathbf{e}_{\ell,\sigma}}$ , then we have  $\varphi_{\ell,\sigma}(\mathbf{v}_i) = \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle$  for all  $1 \leq i \leq d$  and all  $1 \leq \ell \leq r$ .*

In particular, if  $\mathbf{u}_\ell \in P_{\mathbf{e}_{\ell,\sigma}}$  then  $\mathbf{u}_\ell$  is a vertex of this polytope.

*Proof.* The quotient space  $E_{\mathbf{u}_\ell}^\sigma / E_{>\mathbf{u}_\ell}^\sigma$  is identified with the linear subspace  $\text{Span}(\mathbf{e}_{j,\sigma} : \mathbf{u}_j = \mathbf{u}_\ell) \subseteq E$ . Since  $\mathbf{e}_{\ell,\sigma} \in E_{\mathbf{u}_\ell}^\sigma$ , we have  $\mathbf{e}_{\ell,\sigma} \in E^{\mathbf{v}_i}(\langle \mathbf{u}_\ell, \mathbf{v}_i \rangle)$  for  $1 \leq i \leq d$ , so  $\max(j \in \mathbb{Z} : \mathbf{e}_{\ell,\sigma} \in E^{\mathbf{v}_i}(j)) \geq \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle$ . Suppose that, for some index  $i$  satisfying  $1 \leq i \leq d$ , we have  $\max(j \in \mathbb{Z} : \mathbf{e}_{\ell,\sigma} \in E^{\mathbf{v}_i}(j)) > \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle$ . It would follow that  $\mathbf{e}_{\ell,\sigma} \in E_{\mathbf{w}_i + \mathbf{u}_\ell}^\sigma$  and  $\mathbf{e}_{\ell,\sigma} \in E_{>\mathbf{u}_\ell}^\sigma$  which is a contradiction. Therefore, we obtain  $\max(j \in \mathbb{Z} : \mathbf{e}_{\ell,\sigma} \in E^{\mathbf{v}_i}(j)) = \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle$  for all  $1 \leq i \leq d$ . By definition, we have

$$P_{\mathbf{e}_{\ell,\sigma}} := \{\mathbf{u} \in M \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{u}, \mathbf{v}_i \rangle \leq \max(j \in \mathbb{Z} : \mathbf{e}_{\ell,\sigma} \in E^{\mathbf{v}_i}(j)) \text{ for all } 1 \leq i \leq n\}$$

so we conclude that  $\varphi_{\ell,\sigma}(\mathbf{v}_i) = \max(j \in \mathbb{Z} : \mathbf{e}_{\ell,\sigma} \in E^{\mathbf{v}_i}(j)) = \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle$ .  $\square$

Given these preliminaries, we can now give a local description for a global section around the torus-fixed point  $x_\sigma$ . The affine semigroup ring  $\mathbb{C}[\sigma^\vee \cap M]$  is the coordinate ring for the affine open set  $U_\sigma \subset X$  and is isomorphic to the polynomial ring  $\mathbb{C}[y_1, y_2, \dots, y_d]$  where  $y_i := \chi^{\mathbf{w}_i}$  for  $1 \leq i \leq d$ . For any vector  $\mathbf{e} \in E$ , there exists unique scalars  $a_\ell \in \mathbb{C}$ , for all  $1 \leq \ell \leq r$ , such that  $\mathbf{e} = a_1 \mathbf{e}_{1,\sigma} + a_2 \mathbf{e}_{2,\sigma} + \dots + a_r \mathbf{e}_{r,\sigma}$  because  $\mathcal{B}_\sigma = \{\mathbf{e}_{1,\sigma}, \mathbf{e}_{2,\sigma}, \dots, \mathbf{e}_{r,\sigma}\}$  is a basis for  $E$ . By Proposition 1.1, a torus-equivariant global section of  $\mathcal{E}$  has the form  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$ . Hence, the section  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  is given in local coordinates near  $x_\sigma$  by

$$(2.12.3) \quad \sum_{\ell=1}^r a_\ell \left( \mathbf{e}_{\ell,\sigma} \otimes \prod_{i=1}^d y_i^{-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell,\sigma}(\mathbf{v}_i)} \right).$$

Using this local description, we characterize the global generation of a toric vector bundle via the parliament of polytopes.

**Theorem 1.2.** *A toric vector bundle  $\mathcal{E}$  is globally generated if and only if  $\mathbf{u}_\ell \in P_{\mathbf{e}_{\ell,\sigma}}$  for all  $\sigma \in \Sigma(d)$  and all  $\mathbf{u}_\ell \in \mathbf{u}(\sigma)$ .*

*Proof of Theorem 1.2.* As Proposition 1.1 shows,  $\mathcal{E}$  has a torus-equivariant basis of global sections. Hence, the locus in the underlying toric variety  $X$  on which all global sections vanish is closed

and torus-invariant. Since  $X$  is complete, it follows that  $\mathcal{E}$  is globally generated if and only if it is globally generated at every torus-fixed point.

Fix  $\sigma \in \Sigma(d)$  and let  $\mathbf{u}(\sigma) = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ . The vector bundle  $\mathcal{E}$  is globally generated at the torus-fixed point  $x_\sigma$  if and only if the evaluation map

$$\text{ev}_\sigma: H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{E} \otimes \mathcal{O}_X/\mathfrak{m}_{x_\sigma}) \cong \text{Span}(\mathbf{e}_{1,\sigma}, \mathbf{e}_{2,\sigma}, \dots, \mathbf{e}_{r,\sigma})$$

is surjective. A torus-equivariant global section  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  is given in local coordinates near  $x_\sigma$  by (2.12.3). If  $-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell,\sigma}(\mathbf{v}_i) < 0$  for some  $i$ , then we have  $E_{\mathbf{u}}^\sigma \subseteq E_{>\mathbf{u}_\ell}^\sigma$ , which implies that  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  evaluates to zero at  $x_\sigma$ . Otherwise, the evaluation of  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  at  $x_\sigma$  is given by

$$\sum_{\ell=1}^r a_\ell \left( \mathbf{e}_{\ell,\sigma} \otimes \prod_{i=1}^d y_i^{-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell,\sigma}(\mathbf{v}_i)} \right) \Big|_{y_1=y_2=\dots=y_d=0}.$$

This expression is nonzero if and only if there is an index  $\ell$  such that  $-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell,\sigma}(\mathbf{v}_i) = 0$  for all  $1 \leq i \leq d$ . Since  $\varphi_{\ell,\sigma}$  is the support function for the polytope  $P_{\mathbf{e}_{\ell,\sigma}}$ , this is equivalent to saying that  $\mathbf{u} = \mathbf{u}_\ell$  and  $\mathbf{u}_\ell \in P_{\mathbf{e}_{\ell,\sigma}}$ . Hence, a torus-equivariant global section either evaluates to zero at  $x_\sigma$  or equals  $\mathbf{e}_{\ell,\sigma} \otimes \chi^{-\mathbf{u}_\ell}$  and evaluates to  $\mathbf{e}_{\ell,\sigma}$ . Finally, for the evaluation map to be surjective, we need each vector  $\mathbf{e}_{\ell,\sigma}$  to appear.  $\square$

**Corollary 2.13.** *If the toric vector bundle  $\mathcal{E}$  is globally generated, then the polytopes  $P_{\mathbf{e}_{\ell,\sigma}}$  are nonempty for all  $\sigma \in \Sigma(d)$  and all  $\mathbf{e}_{\ell,\sigma} \in \mathcal{B}_\sigma \subseteq \mathbf{u}(\sigma)$ .*  $\square$

*Proof.* This is an immediate consequence of Theorem 1.2.  $\square$

Using Corollary 2.13, we create a low-rank toric vector bundle on  $\mathbb{P}^2$  that is not globally generated; Example 3.4 will show that this low-rank toric vector bundle is also ample.

**Example 2.14.** To describe a second toric vector bundle  $\mathcal{F}$  of rank 3 on  $\mathbb{P}^2$ , we use the notation introduced in Example 2.10. Specifically, the primitive lattice points on the rays in the fan associated to  $\mathbb{P}^2$  are  $\mathbf{v}_1 = (1, 0)$ ,  $\mathbf{v}_2 = (0, 1)$ ,  $\mathbf{v}_3 = (-1, -1)$ , and the maximal cones are  $\sigma_1 = \text{pos}(\mathbf{v}_2, \mathbf{v}_3)$ ,  $\sigma_2 = \text{pos}(\mathbf{v}_1, \mathbf{v}_3)$ ,  $\sigma_3 = \text{pos}(\mathbf{v}_1, \mathbf{v}_2)$ . If  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  denotes the standard basis of  $E = \mathbb{C}^3$ , then the decreasing filtrations defining  $\mathcal{F}$  are

$$E^{\mathbf{v}_1}(j) = \begin{cases} E & \text{if } j \leq -1 \\ \text{Span}(\mathbf{e}_1, \mathbf{e}_2) & \text{if } -1 < j \leq 0 \\ \text{Span}(\mathbf{e}_1) & \text{if } 0 < j \leq 4 \\ 0 & \text{if } 4 < j \end{cases} \quad E^{\mathbf{v}_3}(j) = \begin{cases} E & \text{if } j \leq -1 \\ \text{Span}(\mathbf{e}_3 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2) & \text{if } -1 < j \leq 2 \\ \text{Span}(\mathbf{e}_1 - \mathbf{e}_2) & \text{if } 2 < j \leq 3 \\ 0 & \text{if } 3 < j \end{cases}$$

$$E^{\mathbf{v}_2}(j) = \begin{cases} E & \text{if } j \leq -2 \\ \text{Span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if } -2 < j \leq 0 \\ \text{Span}(\mathbf{e}_3) & \text{if } 0 < j \leq 3 \\ 0 & \text{if } 3 < j \end{cases}.$$

It follows that  $G(\mathcal{M}_{\mathcal{F}}) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_2\}$ , the associated characters and bases are

$$\begin{aligned} \mathbf{u}(\sigma_1) &= \{(-1, -2), (-2, 0), (-2, 3)\} & \mathcal{B}_{\sigma_1} &= \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_2, \mathbf{e}_3\} \\ \mathbf{u}(\sigma_2) &= \{(4, -3), (0, -3), (-1, -1)\} & \mathcal{B}_{\sigma_2} &= \{\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_2\} \\ \mathbf{u}(\sigma_3) &= \{(4, -2), (0, 0), (-1, 3)\} & \mathcal{B}_{\sigma_3} &= \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \end{aligned}$$

and the convex polytopes are

$$\begin{aligned} P_{\mathbf{e}_1} &= \text{Conv}((3, -2), (4, -2), (4, -3)) & P_{\mathbf{e}_1 - \mathbf{e}_2} &= \text{Conv}((-1, -2), (0, -2), (0, -3)) \\ P_{\mathbf{e}_3} &= \text{Conv}((-2, 3), (-1, 3), (-1, 2)) & P_{\mathbf{e}_3 - \mathbf{e}_2} &= \text{Conv}((-2, 0), (-1, 0), (-1, -1)) \\ P_{\mathbf{e}_2} &= \emptyset. \end{aligned}$$

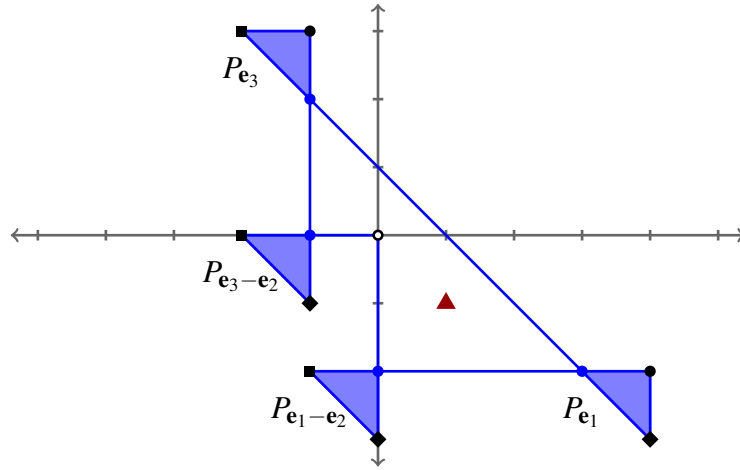


FIGURE 2.14.2. The parliament of polytopes for  $\mathcal{F}$

In Figure 2.14.2, the associated characters are represented by squares, diamonds, and black circles respectively. The polytopes are represented by blue triangles and the other lattice points lying in the polytopes are represented by blue circles. The black circle with white interior represents the unique associated character which does not lie in any of the polytopes. Since  $\mathcal{B}_{\sigma_3} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  and  $P_{\mathbf{e}_2} = \emptyset$ , Corollary 2.13 establishes that  $\mathcal{F}$  is not globally generated.  $\diamond$

**Remark 2.15.** Our diagrams for parliaments of polytopes, such as the one appearing in Figure 2.14.2, have at least some superficial similarities to the twisted polytopes appearing in §6 of [KT]. It would be interesting to develop a more substantive connection.

We close this section with a globally-generated toric vector bundle in which some members of the parliament of polytopes do not correspond to globally-generated line bundles.

**Example 2.16.** To describe our toric vector bundle  $\mathcal{G}$  of rank 2 on the first Hirzebruch surface  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ , we first specify the associated fan. The primitive lattice points on the rays are  $\mathbf{v}_1 = (1, 0)$ ,  $\mathbf{v}_2 = (0, 1)$ ,  $\mathbf{v}_3 = (-1, 1)$ ,  $\mathbf{v}_4 = (0, -1)$ , and the maximal cones are  $\sigma_{1,2} = \text{pos}(\mathbf{v}_1, \mathbf{v}_2)$ ,

$\sigma_{2,3} = \text{pos}(\mathbf{v}_2, \mathbf{v}_3)$ ,  $\sigma_{3,4} = \text{pos}(\mathbf{v}_3, \mathbf{v}_4)$ ,  $\sigma_{1,4} = \text{pos}(\mathbf{v}_1, \mathbf{v}_4)$ . If  $\mathbf{e}_1, \mathbf{e}_2$  denotes the standard basis of  $E = \mathbb{C}^2$ , then the decreasing filtrations defining  $\mathcal{G}$  are

$$E^{\mathbf{v}_1}(j) = \begin{cases} E & \text{if } j \leq -2 \\ \text{Span}(\mathbf{e}_1) & \text{if } -2 < j \leq 4 \\ 0 & \text{if } 4 < j \end{cases} \quad E^{\mathbf{v}_3}(j) = \begin{cases} E & \text{if } j \leq 0 \\ \text{Span}(\mathbf{e}_2) & \text{if } 0 < j \leq 5 \\ 0 & \text{if } 5 < j \end{cases}$$

$$E^{\mathbf{v}_2}(j) = \begin{cases} E & \text{if } j \leq 2 \\ \text{Span}(\mathbf{e}_1) & \text{if } 2 < j \leq 3 \\ 0 & \text{if } 3 < j \end{cases} \quad E^{\mathbf{v}_4}(j) = \begin{cases} E & \text{if } j \leq -1 \\ \text{Span}(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } -1 < j \leq 3 \\ 0 & \text{if } 3 < j \end{cases}$$

It follows that the associated characters and bases are

$$\begin{aligned} \mathbf{u}(\sigma_{1,2}) &= \{(-2, 2), (4, 3)\}, & \mathcal{B}_{\sigma_{1,2}} &= \{\mathbf{e}_2, \mathbf{e}_1\}, \\ \mathbf{u}(\sigma_{2,3}) &= \{(-3, 2), (3, 3)\}, & \mathcal{B}_{\sigma_{2,3}} &= \{\mathbf{e}_2, \mathbf{e}_3\}, \\ \mathbf{u}(\sigma_{3,4}) &= \{(-4, 1), (-3, -3)\}, & \mathcal{B}_{\sigma_{3,4}} &= \{\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}, \\ \mathbf{u}(\sigma_{1,4}) &= \{(-2, -3), (4, 1)\}, & \mathcal{B}_{\sigma_{1,4}} &= \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1\}, \end{aligned}$$

and the convex polytopes are

$$P_{\mathbf{e}_1} = \text{Conv}((1, 1), (3, 3), (4, 3), (4, 1)), \quad P_{\mathbf{e}_1 + \mathbf{e}_2} = \text{Conv}((-3, -3), (-2, -2), (-2, -3)),$$

$$P_{\mathbf{e}_2} = \text{Conv}((-4, 1), (-3, 2), (-2, 2), (-2, 1)).$$

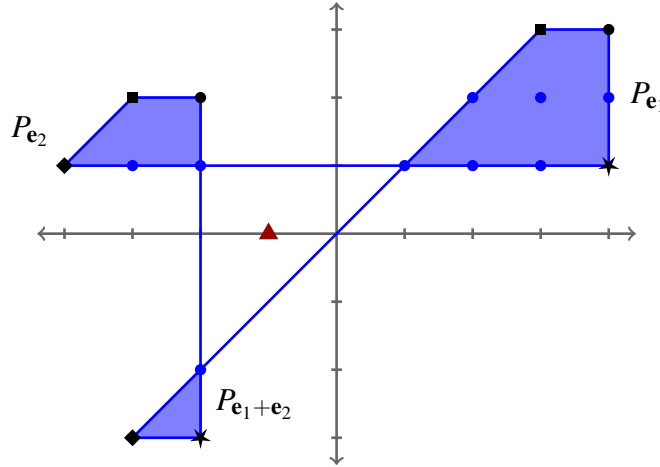


FIGURE 2.16.3. The parliament of polytopes for  $\mathcal{G}$

In Figure 2.16.3, the associated characters are represented by black circles, squares, diamonds, and asterisks respectively. The polytopes are represented by blue regions and the other lattice points lying in the polytopes are represented by blue circles. Theorem 1.2 shows that  $\mathcal{G}$  is globally generated. However, note that the polytopes correspond to the line bundles  $\mathcal{O}_X(4D_1 + 3D_2 - D_4)$ ,  $\mathcal{O}_X(-2D_1 + 2D_2 + 5D_3 - D_4)$ ,  $\mathcal{O}_X(-2D_1 + 2D_2 + 3D_4)$  respectively. The first two line bundles are very ample, but the third is not even globally generated.  $\diamond$

### 3. CONTRASTING NOTIONS OF POSITIVITY

In this section, we analyze some basic positivity phenomenon for toric vector bundles. Specifically, we distinguish the ampleness of a toric vector bundle from other algebraic notions of positivity. We also provide another example of a globally-generated ample toric vector bundle with non-vanishing higher cohomology groups.

Following Definition 6.1.1 in [PAG2], a vector bundle  $\mathcal{E}$  on  $X$  is ample or nef if the line bundle  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  on the projectivized bundle  $\mathbb{P}(\mathcal{E})$  is ample or nef, respectively. Theorem 2.1 in [HMP] provides the key tool for recognizing ample toric vector bundles—a toric vector bundle on a complete toric variety is ample if and only if its restriction to every torus-invariant curve is ample.

**3.1. Restricting to torus-invariant curves.** Consider a torus-invariant curve  $C$  in  $X$  corresponding to the cone  $\tau \in \Sigma(d-1)$ . Since  $X$  is complete, there are two maximal cones  $\sigma$  and  $\sigma'$  in  $\Sigma(d)$  that contain  $\tau$  and  $C \cong \mathbb{P}^1$ . Given two elements  $\mathbf{u}$  and  $\mathbf{u}'$  in  $M$  that agree as linear functionals on  $\tau$ , the toric line bundle  $\mathcal{L}_{\mathbf{u}, \mathbf{u}'}$  on the union  $U_\sigma \cup U_{\sigma'}$  is constructed by gluing  $\mathcal{L}_{\mathbf{u}}|_{U_\sigma}$  and  $\mathcal{L}_{\mathbf{u}'}|_{U_{\sigma'}}$  via the transition function  $\chi^{\mathbf{u}-\mathbf{u}'}$  which is regular and invertible on  $U_\tau$ . If the lattice vector  $\mathbf{v}_\tau \in \sigma$  is dual to the primitive generator of  $\tau^\perp$ , then the line bundle  $\mathcal{L}_{\mathbf{u}, \mathbf{u}'}|_C$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(\langle \mathbf{u}, \mathbf{v}_\tau \rangle D_1 - \langle \mathbf{u}', \mathbf{v}_\tau \rangle D_2)$  where  $D_1$  and  $D_2$  are the prime torus-invariant divisors on  $\mathbb{P}^1$ . Corollary 5.5 and Corollary 5.10 in [HMP] show that the restriction  $\mathcal{E}|_C$  splits equivariantly into a sum of line bundles  $\mathcal{L}_{\mathbf{u}_1, \mathbf{u}'_1}|_C \oplus \mathcal{L}_{\mathbf{u}_2, \mathbf{u}'_2}|_C \oplus \cdots \oplus \mathcal{L}_{\mathbf{u}_r, \mathbf{u}'_r}|_C$  and the pairs  $(\mathbf{u}_i, \mathbf{u}'_i)$  are unique up to reordering. This pairing can be visualized as line segments parallel to  $\tau^\perp$  joining the associated characters in  $\mathbf{u}(\sigma)$  and  $\mathbf{u}(\sigma')$ . Edges in the parliament of polytopes of  $\mathcal{E}$  are contained in such line segments, but the line segments may connect disjoint polytopes. For each individual summand, we have  $\mathcal{L}_{\mathbf{u}, \mathbf{u}'}|_C \cong \mathcal{O}_{\mathbb{P}^1}(a)$  where  $\mathbf{u} - \mathbf{u}'$  is  $a$  times the primitive generator of  $\tau^\perp$  that is positive on  $\sigma$ . Pictorially, the integer  $a$  is the normalized lattice distance between the associated characters in the one-dimensional lattice  $(\tau^\perp + \mathbf{u}) \cap M$ .

To demonstrate these tools, we first reestablish that the tangent bundle on projective space is ample; compare with Remark 2.4 and Example 5.6 in [HMP].

**Example 3.2.** Employing the notation from Example 2.10, the characters associated to the tangent bundle  $\mathcal{T}_{\mathbb{P}^d}$  are  $\mathbf{u}(\sigma_i) = \{\mathbf{w}_1 - \mathbf{w}_i, \mathbf{w}_2 - \mathbf{w}_i, \dots, \mathbf{w}_{i-1} - \mathbf{w}_i, -\mathbf{w}_i, \mathbf{w}_{i+1} - \mathbf{w}_i, \mathbf{w}_{i+2} - \mathbf{w}_i, \dots, \mathbf{w}_d - \mathbf{w}_i\}$  for  $1 \leq i \leq d$ , and  $\mathbf{u}(\sigma_{d+1}) = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}$ . On the torus-invariant curve  $C_{i,j}$  corresponding to the cone  $\tau_{i,j} := \sigma_i \cap \sigma_j \in \Sigma(d-1)$  where  $1 \leq i < j \leq d$ , the characters in  $\mathbf{u}(\sigma_i)$  and  $\mathbf{u}(\sigma_j)$  are paired as follows:  $(-\mathbf{w}_i, \mathbf{w}_i - \mathbf{w}_j)$ ,  $(\mathbf{w}_j - \mathbf{w}_i, -\mathbf{w}_j)$ , and  $(\mathbf{w}_k - \mathbf{w}_i, \mathbf{w}_k - \mathbf{w}_j)$  for all  $k \neq i$  or  $j$ . Thus, we deduce that  $\mathcal{T}_{\mathbb{P}^d}|_{C_{i,j}} \cong \mathcal{O}_{\mathbb{P}^1}(D_1 + D_2) \oplus (\bigoplus_{j=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(D_2)) \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus (\bigoplus_{j=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(1))$ . A similar calculation for the curve  $C_{i,d+1}$ , which corresponds to the cone  $\tau_{i,d+1} := \sigma_i \cap \sigma_{d+1} \in \Sigma(d-1)$  where  $1 \leq i \leq d$ , yields  $\mathcal{T}_{\mathbb{P}^d}|_{C_{i,d+1}} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus (\bigoplus_{j=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(1))$ . Since the restriction to every torus-invariant curve is ample, we conclude that  $\mathcal{T}_{\mathbb{P}^d}$  is ample.  $\diamond$

With these tools, we can also prove directly that the cotangent bundle on a smooth toric variety is never ample; compare with §6.3B in [PAG2].

**Example 3.3.** Let  $\Omega_X$  denote the cotangent bundle on a smooth toric variety  $X$ , and let  $\Sigma$  be the fan of  $X$ . Identifying the fibre  $E$  over the identity of the torus with  $M \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^d$  as done in §2.3.5 in [Kly], the decreasing filtrations for  $\Omega_X$  are

$$E^{\mathbf{v}_i}(j) = \begin{cases} E & \text{if } j \leq -1 \\ \mathbf{v}_i^\perp & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases} \quad \text{for all } 1 \leq i \leq n.$$

Consider two adjacent cones  $\sigma, \sigma' \in \Sigma(d)$ . Since  $X$  is smooth, we have  $\sigma = \text{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$  where  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$  is a basis for  $N$ . We may assume that  $\sigma' = \text{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d-1}, \mathbf{v}_{d+1})$  where  $\mathbf{v}_{d+1} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_{d-1}\mathbf{v}_{d-1} - \mathbf{v}_d$  for some  $a_j \in \mathbb{Z}$ . If  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d \in M$  form the dual basis to  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$ , then the associated characters are  $\mathbf{u}(\sigma) = \{-\mathbf{w}_1, -\mathbf{w}_2, \dots, -\mathbf{w}_d\}$  and  $\mathbf{u}(\sigma') = \{-\mathbf{w}_1 - a_1\mathbf{w}_d, -\mathbf{w}_2 - a_2\mathbf{w}_d, \dots, -\mathbf{w}_{d-1} - a_{d-1}\mathbf{w}_d, \mathbf{w}_d\}$ . Along the torus-invariant curve  $C$  corresponding to the cone  $\tau = \sigma \cap \sigma' \in \Sigma(d-1)$ , the characters are paired as follows:  $(-\mathbf{w}_d, \mathbf{w}_d)$  and  $(-\mathbf{w}_i, -\mathbf{w}_i - a_i\mathbf{w}_d)$  for  $1 \leq i \leq d-1$ . Therefore, we obtain  $\Omega_X|_C \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus (\bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(a_i))$  which implies that  $\Omega_X$  is not ample.  $\diamond$

More significantly, we next exhibit an ample toric vector bundle on a smooth toric variety that is not globally generated. In particular, this supersedes Examples 4.15–4.17 in [HMP] and answers the second part of Question 7.5 in [HMP].

**Example 3.4.** Consider the toric vector bundle  $\mathcal{F}$  on  $\mathbb{P}^2$  appearing in Example 2.14. Having already established that  $\mathcal{F}$  is not globally generated, it remains to show that  $\mathcal{F}$  is ample. Let  $C_k$  denote the torus-invariant curve in  $\mathbb{P}^2$  corresponding to the cone  $\tau_{i,j} := \sigma_i \cap \sigma_j \in \Sigma(d-1)$  where  $\{i, j, k\} = \{1, 2, 3\}$ . From the line segments in Figure 2.14.2 joining black circles to diamonds, we see that the characters in  $\mathbf{u}(\sigma_2)$  and  $\mathbf{u}(\sigma_3)$  are paired on  $C_1$  as  $((-1, 3), (-1, -1)), ((0, 0), (0, -3)), ((4, -2), (4, -3))$ , so we obtain

$$\mathcal{F}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(3D_1 + D_2) \oplus \mathcal{O}_{\mathbb{P}^1}(3D_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2D_1 + 3D_2) \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Similar calculations give  $\mathcal{F}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$  and  $\mathcal{F}|_{C_3} \cong \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . Since the restriction to every torus-invariant curve is ample, the toric vector bundle  $\mathcal{F}$  is ample.  $\diamond$

The vector bundle  $\mathcal{F}$  has minimal rank among all ample toric vector bundles on  $\mathbb{P}^2$  that are not globally generated. More than that, the ensuing proposition proves that, for low-rank toric vector bundles on  $\mathbb{P}^d$ , nef is equivalent to globally generated.

**Proposition 3.5.** *If  $\mathcal{E}$  is a toric vector bundle on  $\mathbb{P}^d$  with rank at most  $d$ , then  $\mathcal{E}$  is globally generated if and only if it is nef.*

*Proof.* As follows from Example 1.4.5 in [PAG1], every globally generated vector bundle is nef, so it suffices to prove the converse implication. Moreover, a line bundle on a complete toric variety is nef if and only if it is globally generated; see Theorem 6.3.13 in [CLS]. Hence, the proposition follows immediately when  $\mathcal{E}$  splits as a direct sum of line bundles. If the rank of  $\mathcal{E}$  is less than  $d$ , then Corollary 3.5 in [Kan] or Corollary 6.1.5 in [Kly] imply that  $\mathcal{E}$  splits into a direct sum of line bundles. Therefore, we may assume that  $\mathcal{E}$  is indecomposable and has rank equal to  $d$ .



Under these hypotheses, Theorem 4.6 in [Kan] establishes that  $\mathcal{E}$  is isomorphic to either  $\mathcal{Q}(\ell)$  or  $\mathcal{Q}^*(\ell)$  for some  $\ell \in \mathbb{Z}$ , where  $\mathcal{Q}$  is defined by the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^d} \xrightarrow{\begin{bmatrix} y_1^{a_1} & y_2^{a_2} & \cdots & y_{d+1}^{a_{d+1}} \end{bmatrix}} \bigoplus_{k=1}^{d+1} \mathcal{O}_{\mathbb{P}^d}(a_k D_k) \longrightarrow \mathcal{Q} \longrightarrow 0,$$

$a_1, a_2, \dots, a_{d+1}$  are positive integers, and  $D_1, D_2, \dots, D_{d+1}$  are the torus-invariant divisors on  $\mathbb{P}^d$ . Using the notation from Example 3.2, let  $C_{i,j}$  denote the torus-invariant curve corresponding to the cone  $\tau_{i,j} = \sigma_i \cap \sigma_j \in \Sigma(d-1)$  where  $1 \leq i < j \leq d+1$ . Restricting the short exact sequence to the curve  $C_{i,j}$ , we obtain  $\mathcal{Q}|_{C_{i,j}} \cong \mathcal{O}_{\mathbb{P}^1}(a_i + a_j) \oplus (\bigoplus_{k=1, k \neq i,j}^{d+1} \mathcal{O}_{\mathbb{P}^1}(a_k))$ .

If  $\mathcal{E} = \mathcal{Q}(\ell)$  and  $\mathcal{E}$  is nef, then we have  $a_k + \ell \geq 0$  for all  $1 \leq k \leq d+1$  which means that the vector bundle  $\mathcal{S} := \bigoplus_{k=1}^{d+1} \mathcal{O}_{\mathbb{P}^d}(a_k + \ell)$  is globally generated. Since  $\mathcal{E}$  is a quotient of  $\mathcal{S}$ , we conclude that  $\mathcal{E}$  is also globally generated; see Example 6.1.4 in [PAG2]. If  $\mathcal{E} = \mathcal{Q}^*(\ell)$  and  $\mathcal{E}$  is nef, then we have  $\ell - a_k \geq 0$  for all  $1 \leq k \leq d+1$  and  $\ell - a_i - a_j \geq 0$  for all  $1 \leq i < j \leq d+1$ . The functorial properties of the dual imply that  $\mathcal{Q}^*(\ell) \hookrightarrow \bigoplus_{k=1}^{d+1} \mathcal{O}_{\mathbb{P}^d}(\ell - a_k)$  and  $\mathcal{Q}^*(\ell) \cong (\bigwedge^{d-1} \mathcal{Q}^*(\ell))^* \otimes \det(\mathcal{Q}^*(\ell))$ . It follows that  $\mathcal{E}$  is quotient of the vector bundle  $\mathcal{S}' := (\bigwedge^{d-1} (\bigoplus_{k=1}^{d+1} \mathcal{O}_{\mathbb{P}^d}(\ell - a_k)))^* \otimes \det(\mathcal{Q}^*(\ell))$ . Since  $\bigwedge^{d-1} (\bigoplus_{k=1}^{d+1} \mathcal{O}_{\mathbb{P}^d}(\ell - a_k)) \cong \bigoplus_{1 \leq k_1 < k_2 < \dots < k_{d-1} \leq d+1} \mathcal{O}_{\mathbb{P}^d}((d-1)\ell - a_{k_1} - a_{k_2} - \dots - a_{k_{d-1}})$  and  $\det(\mathcal{Q}^*(\ell)) \cong \mathcal{O}_{\mathbb{P}^d}(d\ell - a_1 - a_2 - \dots - a_{d+1})$ , we see that  $\mathcal{S}'$  is a direct sum of line bundles of the form  $\mathcal{O}_{\mathbb{P}^d}(\ell - a_j - a_k)$  which implies that both  $\mathcal{S}'$  and  $\mathcal{E}$  are globally generated.  $\square$

To compliment Examples 4.9–4.10 in [HMP], we end this section by illustrating that the higher cohomology groups of a globally-generated ample toric vector bundle on a smooth toric variety may be nonzero.

**Example 3.6.** Consider the globally-generated toric vector bundle  $\mathcal{G}$  appearing in Example 2.16. Restricting to the torus-invariant curves gives  $\mathcal{G}|_{C_1} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ ,  $\mathcal{G}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ ,  $\mathcal{G}|_{C_3} \cong \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ ,  $\mathcal{G}|_{C_4} \cong \mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ , and shows that  $\mathcal{G}$  is ample. Furthermore, Theorem 4.2.1 in [Kly] establishes that the equivariant Euler characteristic of  $\mathcal{G}$  is

$$\begin{aligned} \chi(\mathcal{G}) &= \sum_i (-1)^i \dim H^i(X, \mathcal{G})_{\mathbf{u}} \cdot t^{\mathbf{u}} = \frac{t_1^{-2}t_2^2 + t_1^4t_2^3}{(1-t_1)(1-t_2)} + \frac{t_1^{-3}t_2^2 + t_1^3t_2^3}{(1-t_1)(1-t_1^{-1}t_2^{-1})} + \frac{t_1^{-4}t_2 + t_1^{-3}t_2^{-3}}{(1-t_1)(1-t_1t_2)} + \frac{t_1^{-2}t_2^{-3} + t_1^4t_2}{(1-t_1^{-1})(1-t_2)} \\ &= t_1^4t_2^3 + t_1^4t_2^2 + t_1^4t_2 + t_1^3t_2^3 + t_1^3t_2^2 + t_1^3t_2 + t_1^2t_2^2 + t_1^2t_2 + t_1t_2 \\ &\quad - t_1^{-1} + t_1^{-2}t_2^2 + t_1^{-2}t_2 + t_1^{-2}t_2^{-2} + t_1^{-2}t_2^{-3} + t_1^{-3}t_2^2 + t_1^{-3}t_2 + t_1^{-3}t_2^{-3} + t_1^{-4}t_2, \end{aligned}$$

so we have  $H^1(X, \mathcal{G})_{(-1,0)} \neq 0$ . Using Theorem 4.1.1 in [Kly], a longer calculation confirms that  $H^1(X, \mathcal{G})_{\mathbf{u}} \cong \mathbb{C}$  when  $\mathbf{u} = (-1, 0)$  and  $H^1(X, \mathcal{G})_{\mathbf{u}} = 0$  when  $\mathbf{u} \neq (-1, 0)$ . In Figure 2.16.3, the red triangle represents the unique character for which the higher cohomology groups do not vanish.  $\diamond$

**Remark 3.7.** Using the techniques from Example 3.6 or Example 4.3.5 in [Kly], we see that  $H^1(\mathbb{P}^2, \mathcal{F})_{\mathbf{u}} \neq 0$  where  $\mathbf{u} = (1, -1)$  and  $\mathcal{F}$  is the toric vector bundle appearing in Example 2.14. In Figure 2.14.2, the red triangle represents the unique character for which the higher cohomology groups do not vanish.

#### 4. HIGHER-ORDER JETS

This section relates positivity of higher-order jets to properties of the associated parliament of polytopes. In particular, we determine which results for jets of line bundles on smooth toric varieties extend to higher-rank toric vector bundles.

**4.1. Positivity properties of jets.** Fix  $k \in \mathbb{N}$ . A vector bundle  $\mathcal{E}$  *separates  $k$ -jets* if, for every closed point  $x \in X$  with maximal ideal  $\mathfrak{m}_x \subseteq \mathcal{O}_X$ , the map  $J_x^k: H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x^{k+1})$ , which evaluates a global section and its derivatives of order at most  $k$  at  $x$ , is surjective; compare with Definition 5.1.15 in [PAG1]. When  $X$  is a toric variety, this map is torus-equivariant, because differentiation is  $\mathbb{C}$ -linear. As a special case, we see that a vector bundle separates 0-jets if and only if it is globally generated. A vector bundle that separates  $k$ -jets is also called  *$k$ -jet spanned*.

As a stronger attribute, we say that a vector bundle  $\mathcal{E}$  is  *$k$ -jet ample* if, for all distinct closed points  $x_1, x_2, \dots, x_t \in X$  and all positive integers  $k_1, k_2, \dots, k_t$  satisfying  $\sum_{i=1}^t k_i = k + 1$ , the natural map  $\psi: H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/(\mathfrak{m}_{x_1}^{k_1} \cdot \mathfrak{m}_{x_2}^{k_2} \cdots \mathfrak{m}_{x_t}^{k_t})) = \bigoplus_{i=1}^t H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_{x_i}^{k_i})$  is surjective. Hence, a  $k$ -jet ample vector bundle does separate  $k$ -jets, and a vector bundle separates 0-jets if and only if it is 0-jet ample. Proposition 4.2 in [BDS] proves that every 1-jet ample vector bundle on a smooth projective variety is very ample, and Example 4.3 in [BDS] shows that the converse does not always hold. If  $0 \leq \ell \leq k$ , then a vector bundle that separates  $k$ -jets also separates  $\ell$ -jets, and a vector bundle that is  $k$ -jet ample is also  $\ell$ -jet ample.

We start by placing ampleness into this hierarchy of positivity properties on a smooth toric variety.

**Lemma 4.2.** *Every toric vector bundle that separates 1-jets is ample.*

*Proof.* Let  $\mathcal{E}$  be toric vector bundle that separates 1-jets. For any torus-invariant curve  $C$ , the restriction  $\mathcal{E}|_C$  separates 1-jets and splits equivariantly into sum of line bundles. For a line bundle on a toric variety, Theorem 4.2 in [DiR] shows that separating 1-jets is equivalent to being ample. Hence, if  $\mathcal{E}|_C \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ , then each line bundle  $\mathcal{O}_{\mathbb{P}^1}(a_i)$  is ample. Therefore, the restriction to every torus-invariant curve is ample, which ensures that  $\mathcal{E}$  is ample.  $\square$

We next characterize the toric vector bundles that separate  $k$ -jets by generalizing Theorem 1.2.

**Theorem 4.3.** *A toric vector bundle  $\mathcal{E}$  separates  $k$ -jets if and only if  $\mathbf{u}_\ell \in \mathbf{P}_{\ell, \sigma}$ , for all  $\sigma \in \Sigma(d)$  and all  $\mathbf{u}_\ell \in \mathbf{u}(\sigma)$ , and the edges emanating from  $\mathbf{u}_\ell$  have lattice length at least  $k$ .*

*Proof.* The locus in the underlying variety  $X$ , on which  $H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x^{k+1})$  is not surjective, is closed and torus-invariant. Since  $X$  is complete, it follows that  $\mathcal{E}$  separates  $k$ -jets if and only if it separates  $k$ -jets at the torus-fixed points.

Fix  $\sigma \in \Sigma(d)$  and let  $\mathbf{u}(\sigma) = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ . The vector bundle  $\mathcal{E}$  separates  $k$ -jets at the torus-fixed point  $x_\sigma$  if and only if the natural map

$$J_{x_\sigma}^k: H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_{x_\sigma}^{k+1}) \cong \text{Span}(\mathbf{e}_{1, \sigma}, \mathbf{e}_{2, \sigma}, \dots, \mathbf{e}_{r, \sigma}) \otimes \mathbb{C}^{\binom{k+d}{d}}$$

is surjective, where the standard basis for vector space  $\mathbb{C}^{\binom{k+d}{d}}$  corresponds to the partial derivatives of order less than  $k$ . A torus-equivariant global section  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  is given in local coordinates near

$x_\sigma$  by (2.12.3). If  $-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell, \sigma}(\mathbf{v}_i) < 0$  for some  $i$ , then we have  $E_{\mathbf{u}}^\sigma \subseteq E_{>\mathbf{u}_\ell}^\sigma$ , which implies that  $J_{x_\sigma}^k(\mathbf{e} \otimes \chi^{-\mathbf{u}})$  is zero. Otherwise, the map  $J_{x_\sigma}^k$  sends  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  to the first  $k$  terms of the Taylor expansion about  $x_\sigma$ . Hence, for  $\mathbf{m} \in \mathbb{N}^d$  satisfying  $|\mathbf{m}| \leq k$ , the  $\mathbf{m}$ -th component of  $J_{x_\sigma}^k(\mathbf{e} \otimes \chi^{-\mathbf{u}})$  is given in local coordinates by

$$\sum_{\ell=1}^r a_\ell \left( \mathbf{e}_{\ell, \sigma} \otimes \frac{1}{m_1! m_2! \cdots m_d!} \frac{\partial^{m_1+m_2+\cdots+m_d}}{\partial^{m_1} y_1 \partial^{m_2} y_2 \cdots \partial^{m_d} y_d} \left( \prod_{i=1}^d y_i^{-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell, \sigma}(\mathbf{v}_i)} \right) \right) \Big|_{y_1=y_2=\cdots=y_d=0}.$$

This expression is nonzero if and only if there is an index  $\ell$  such that  $-\langle \mathbf{u}, \mathbf{v}_i \rangle + \varphi_{\ell, \sigma}(\mathbf{v}_i) = m_\ell$  for all  $1 \leq i \leq d$ . Since  $\varphi_{\ell, \sigma}$  is the support function for the polytope  $P_{\mathbf{e}_{\ell, \sigma}}$ , this is equivalent to saying that  $\mathbf{u} = \mathbf{u}_\ell + \mathbf{m}$  and  $\mathbf{u}_\ell + \mathbf{m} \in P_{\mathbf{e}_{\ell, \sigma}}$ . Hence, the  $\mathbf{m}$ -th component of  $J_{x_\sigma}^k(\mathbf{e} \otimes \chi^{-\mathbf{u}})$  is either zero or  $\mathbf{e}_{\ell, \sigma}$  and  $\mathbf{e} \otimes \chi^{-\mathbf{u}} = \mathbf{e}_{\ell, \sigma} \otimes \chi^{-\mathbf{u}_\ell - \mathbf{m}}$ . For the map  $J_{x_\sigma}^k$  to be surjective, we need each vector  $\mathbf{e}_{\ell, \sigma}$  to appear in each component. By convexity, this characterization is equivalent to requiring that the edges through the vertex  $\mathbf{u}_i$  have lattice length at least  $k$ .  $\square$

The following two corollaries provide alternative criteria for separating jets. For a maximal cone  $\sigma = \text{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$ , let  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$  denote the minimal generators of the dual cone  $\sigma^\vee$ , and set  $\Delta_{\ell, \sigma} := \text{Conv}(\mathbf{u}_\ell, \mathbf{u}_\ell + \mathbf{w}_1, \mathbf{u}_\ell + \mathbf{w}_2, \dots, \mathbf{u}_\ell + \mathbf{w}_d)$ . In other words,  $\Delta_{\ell, \sigma}$  is the lattice simplex with vertex  $\mathbf{u}_\ell$  and edges parallel to the generators of  $\sigma^\vee$ .

**Corollary 4.4.** *A toric vector bundle  $\mathcal{E}$  separates 1-jets if and only if the polytopes  $P_{\mathbf{e}_{\ell, \sigma}}$  are full dimensional for all  $\sigma \in \Sigma(d)$  and all  $\mathbf{e}_{\ell, \sigma} \in \mathcal{B}_\sigma$ .*

*Proof.* Lemma 2.12 implies that, whenever the polytope  $P_{\mathbf{e}_{\ell, \sigma}}$  is nonempty, the character  $\mathbf{u}_\ell$  is a vertex of  $P_{\mathbf{e}_{\ell, \sigma}}$ . It follows that  $P_{\mathbf{e}_{\ell, \sigma}}$  is  $d$ -dimensional if and only if it contains  $\Delta_{\ell, \sigma}$ . Therefore, Theorem 4.3 establishes the required equivalence.  $\square$

**Corollary 4.5.** *A toric vector bundle  $\mathcal{E}$  separates  $k$ -jets at the torus-fixed point  $x_\sigma \in X$  if and only if the map induced by  $J_{x_\sigma}^k$  from  $H^0(X, \mathcal{E})_{\mathbf{u}'} \cap E_{\mathbf{u}_\ell}^\sigma$  to  $E_{\mathbf{u}_\ell}^\sigma / E_{>\mathbf{u}_\ell}^\sigma$  is surjective for all  $\mathbf{u}_\ell \in \mathbf{u}(\sigma)$  and all  $\mathbf{u}' \in \mathbf{u}_\ell - k \cdot \Delta_{\ell, \sigma}$ .*

*Proof.* The proof of Theorem 4.3 shows that  $\mathcal{E}$  separates  $k$ -jets at the torus-fixed point  $x_\sigma \in X$  if and only if  $k \cdot \Delta_{\ell, \sigma} \subseteq P_{\mathbf{e}_{\ell, \sigma}}$  for all  $\mathbf{u}_\ell \in \mathbf{u}(\sigma)$ . Since  $E_{\mathbf{u}_\ell}^\sigma / E_{>\mathbf{u}_\ell}^\sigma \cong \text{Span}(\mathbf{e}_{j, \sigma} : \mathbf{u}_j = \mathbf{u}_\ell)$ , this is the same as saying that, for  $\mathbf{u}_j = \mathbf{u}_\ell$  and  $\mathbf{u}' \in \mathbf{u}_\ell - k \cdot \Delta_{\ell, \sigma}$ , the section  $\mathbf{e}_{j, \sigma} \otimes \chi^{-\mathbf{u}'}$  has a nonzero image under  $J_{x_\sigma}^k$ , or equivalently the linear space  $H^0(X, \mathcal{E})_{\mathbf{u}'} \cap E_{\mathbf{u}_\ell}^\sigma$  surjects onto  $E_{\mathbf{u}_\ell}^\sigma / E_{>\mathbf{u}_\ell}^\sigma$ .  $\square$

With Theorem 4.3, we easily verify that the tangent bundle on projective space separates 1-jets.

**Example 4.6.** As computed in Example 2.10, the parliament of polytopes for the tangent bundle  $\mathcal{T}_{\mathbb{P}^d}$  consists of  $P_{\mathbf{v}_i} = \text{Conv}(\mathbf{0}, \mathbf{w}_i - \mathbf{w}_1, \mathbf{w}_i - \mathbf{w}_2, \dots, \mathbf{w}_i - \mathbf{w}_{i-1}, \mathbf{w}_i, \mathbf{w}_i - \mathbf{w}_{i+1}, \mathbf{w}_i - \mathbf{w}_{i+1}, \dots, \mathbf{w}_i - \mathbf{w}_d)$  for  $1 \leq i \leq d$ , and  $P_{\mathbf{v}_{d+1}} = \text{Conv}(\mathbf{0}, -\mathbf{w}_1, -\mathbf{w}_2, \dots, -\mathbf{w}_d)$ . Hence, the edges in each polytope have normalized length 1 and generate the appropriate maximal cone, so  $\mathcal{T}_{\mathbb{P}^d}$  separates 1-jets.  $\diamond$

Since Example 2.14 already exhibits an ample toric vector bundle that is not globally generated, the converse to Lemma 4.2 is obviously false. To sharpen this distinction, we present an ample toric vector bundle that is globally generated but does not separate 1-jets.

**Example 4.7.** Using the notation from Example 2.14 and Example 3.4, consider the toric vector bundle  $\mathcal{H}$  of rank 3 on  $\mathbb{P}^2$  defined by the following decreasing filtrations:

$$E^{v_1}(j) = \begin{cases} E & \text{if } j \leq -2 \\ \text{Span}(\mathbf{e}_1, \mathbf{e}_2) & \text{if } -2 < j \leq -1 \\ \text{Span}(\mathbf{e}_1) & \text{if } -1 < j \leq 2 \\ 0 & \text{if } 2 < j \end{cases} \quad E^{v_3}(j) = \begin{cases} E & \text{if } j \leq 1 \\ \text{Span}(\mathbf{e}_3 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2) & \text{if } 1 < j \leq 3 \\ \text{Span}(\mathbf{e}_1 - \mathbf{e}_2) & \text{if } 3 < j \leq 4 \\ 0 & \text{if } 4 < j \end{cases}$$

$$E^{v_2}(j) = \begin{cases} E & \text{if } j \leq -2 \\ \text{Span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if } -2 < j \leq 0 \\ \text{Span}(\mathbf{e}_3) & \text{if } 0 < j \leq 2 \\ 0 & \text{if } 2 < j \end{cases}.$$

It follows that the associated characters and bases are

$$\begin{aligned} \mathbf{u}(\sigma_1) &= \{(-2, -2), (-3, 0), (-3, 2)\} & \mathcal{B}_{\sigma_1} &= \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_2, \mathbf{e}_3\} \\ \mathbf{u}(\sigma_2) &= \{(2, -3), (-1, -3), (-2, -1)\} & \mathcal{B}_{\sigma_2} &= \{\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_2\} \\ \mathbf{u}(\sigma_3) &= \{(2, -2), (-1, 0), (-2, 2)\} & \mathcal{B}_{\sigma_3} &= \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \end{aligned}$$

and the convex polytopes are

$$\begin{aligned} P_{\mathbf{e}_1} &= \text{Conv}((1, -2), (2, -2), (2, -3)) & P_{\mathbf{e}_1 - \mathbf{e}_2} &= \text{Conv}((-2, -2), (-1, -2), (-1, -3)) \\ P_{\mathbf{e}_3} &= \text{Conv}((-3, 2), (-2, 2), (-2, 1)) & P_{\mathbf{e}_3 - \mathbf{e}_2} &= \text{Conv}((-3, 0), (-2, 0), (-2, -1)) \\ P_{\mathbf{e}_2} &= \text{Conv}((-1, 0)). \end{aligned}$$

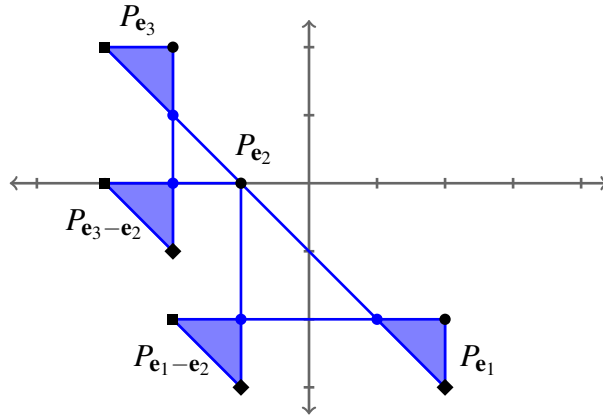


FIGURE 4.7.4. The parliament of polytopes for  $\mathcal{H}$

In Figure 4.7.4, the associated characters are represented by black squares, diamonds, and circles respectively. The polytopes are represented by blue regions and the other lattice points lying in the polytopes are represented by blue circles. Using Corollary 2.13, we see that  $\mathcal{H}$  is globally generated. In contrast, Theorem 4.3 implies that  $\mathcal{H}$  does not separate 1-jets because  $P_{\mathbf{e}_2}$  is simply a point. Lastly, restricting to the torus-invariant curves gives  $\mathcal{H}|_{C_1} \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ ,

$\mathcal{H}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ , and  $\mathcal{H}|_{C_3} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ , so the toric vector bundle is ample.  $\diamond$

On a smooth projective variety, being 1-jet ample is generally a stronger condition than separating 1-jets, as Example 2.3 in [LM] and Example 4.6 in [Lan] demonstrate for line bundles. For line bundles on a smooth complete toric variety, these conditions are equivalent; see [DiR]. Extending this result, we prove that these conditions are equivalent for toric vector bundles on a smooth complete toric variety.

**Theorem 4.8.** *A toric vector bundle separates  $k$ -jets if and only if it is  $k$ -jet ample.*

*Proof.* It suffices to show that every toric vector bundle  $\mathcal{E}$  which separates  $k$ -jets is  $k$ -jet ample. The locus in the toric variety  $\prod_{i=1}^t X$ , on which  $H^0(X, \mathcal{E}) \rightarrow \bigoplus_{i=1}^t H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_{x_i}^{k_i})$  is not surjective, is closed and torus-invariant. Since  $X$  is complete, it follows that  $\mathcal{E}$  is  $k$ -jet ample if and only if it is  $k$ -jet ample at the torus-fixed points. Thus, it is enough to prove that, for all distinct torus-fixed points  $x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_t}$  and all positive integers  $k_1, k_2, \dots, k_t$  satisfying  $\sum_{i=1}^t k_i = k + 1$ , the map  $\psi: H^0(X, \mathcal{E}) \rightarrow \bigoplus_{i=1}^t H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_{x_{\sigma_i}}^{k_i})$  is surjective.

Since  $\mathcal{E}$  separates  $k_i$ -jets, for all  $k_i \leq k$ , the map  $\psi$  surjects onto each individual summand. Fix an index  $i$  satisfying  $1 \leq i \leq t$  and consider a torus-equivariant global section  $\mathbf{e} \otimes \chi^{-\mathbf{u}}$  such that  $0 \neq J_{x_{\sigma_i}}^{k_i-1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) \in H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_{x_{\sigma_i}}^{k_i})$ . To prove that  $\psi$  is surjective, it is enough to show that  $J_{x_{\sigma_j}}^{k_j-1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) = 0$  for all  $j \neq i$ . As in the proof of Theorem 4.3, the hypothesis  $0 \neq J_{x_{\sigma_i}}^{k_i-1}(\mathbf{e} \otimes \chi^{-\mathbf{u}})$  implies that there exists  $\mathbf{u}_\ell \in \mathbf{u}(\sigma_i)$  and  $\mathbf{m} \in \mathbb{N}^d$  such that  $|\mathbf{m}| \leq k_i - 1$ ,  $\mathbf{e} = \mathbf{e}_{\ell, \sigma_i}$ , and  $\mathbf{u} = \mathbf{u}_\ell + \mathbf{m} \in P_{\mathbf{e}}$ . Since we have  $P_{\mathbf{e}_{\ell, \sigma_i}} \neq \emptyset$ , Lemma 2.12 confirms that  $\mathbf{u}_\ell$  as a vertex of the polytope  $P_{\mathbf{e}}$ , and  $\mathbf{u} \in (k_i - 1) \cdot \Delta_{\ell, \sigma_i} \subseteq P_{\mathbf{e}}$ . If  $J_{x_{\sigma_j}}^{k_j-1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) \neq 0$ , then there also exists  $\mathbf{u}_{\ell'} \in \mathbf{u}(\sigma_j)$  such that  $\mathbf{e} = \mathbf{e}_{\ell', \sigma_j}$ ,  $\mathbf{u}_{\ell'}$  is a vertex of  $P_{\mathbf{e}}$ , and  $\mathbf{u} \in (k_j - 1) \cdot \Delta_{\ell', \sigma_j}$ . We have  $\mathbf{u}_{\ell'} \neq \mathbf{u}_\ell$  because  $j \neq i$ . As  $\mathcal{E}$  separates  $k$ -jets at  $x_{\sigma_i}$ , Theorem 4.3 implies that lattice length of each edge in  $P_{\mathbf{e}}$  emanating from the vertex  $\mathbf{u}_\ell$  is at least  $k$ . Since  $k_i + k_j - 2 \leq k - 1$ , the convexity of  $P_{\mathbf{e}}$  guarantees that  $\emptyset = (k_i - 1) \cdot \Delta_{\ell, \sigma_i} \cap (k_j - 1) \cdot \Delta_{\ell', \sigma_j} \subset P_{\mathbf{e}}$  which is a contradiction. Therefore, we conclude that  $J_{x_{\sigma_i}}^{k_i-1}(\mathbf{e} \otimes \chi^{-\mathbf{u}}) = 0$  and  $\psi$  is surjective.  $\square$

For a line bundle on a smooth toric variety, Theorem 4.2 in [DiR] establishes that separating 1-jets is equivalent to being very ample. As a final result, we generalize this equivalence to higher-rank toric vector bundles on a smooth toric variety.

**Theorem 4.9.** *A toric vector bundle separates 1-jets if and only if it is very ample.*

*Proof.* It suffices to show that every very ample toric vector bundle  $\mathcal{E}$  separates 1-jets at the torus-fixed points. Let  $X$  be the underlying smooth toric variety determined by the fan  $\Sigma$ . Fix a maximal cone  $\sigma_0 \in \Sigma(d)$  and consider the blowing up  $\pi: X' \rightarrow X$  of  $X$  at  $x_{\sigma_0}$  with exceptional divisor  $D_0 := \pi^{-1}(x_{\sigma_0})$ . Since  $\mathcal{E}$  is very ample, Corollary 1 in [BSS] establishes that the toric vector bundle  $\mathcal{E}' := \pi^*(\mathcal{E}) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-D_0)$  is globally generated.

To complete the proof, we relate the parliament of polytopes for  $\mathcal{E}'$  and  $\mathcal{E}$ . First, we describe the underlying fan for  $X'$ . Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be the primitive lattice vectors generating the rays in  $\Sigma$ . By reordering these rays if necessary, we may assume that  $\sigma_0 = \text{pos}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)$ . If  $\mathbf{v}_0 := \mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_d$  and  $\Sigma'$  is the fan of  $X'$ , then the primitive lattice vectors generating the rays in  $\Sigma'$  are  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ , and the maximal cones are  $\Sigma'(d) = (\Sigma(d) \setminus \sigma_0) \cup \{\sigma_1, \sigma_2, \dots, \sigma_d\}$  where  $\sigma_i := \text{pos}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \dots, \mathbf{v}_d)$  for  $1 \leq i \leq d$ ; compare with Example 3.1.15 in [CLS].

We next specify the linear invariants which determine the toric vector bundle  $\mathcal{E}'' := \pi^*(\mathcal{E})$  on  $X'$ . The characters associated to  $\mathcal{E}''$  are  $\mathbf{u}_{\mathcal{E}''}(\sigma') = \mathbf{u}_{\mathcal{E}}(\sigma')$  for all  $\sigma' \in \Sigma(d) \setminus \sigma_0$  and  $\mathbf{u}_{\mathcal{E}''}(\sigma_i) = \mathbf{u}_{\mathcal{E}}(\sigma_0)$  for all  $1 \leq i \leq d$ . The compatible decreasing filtrations corresponding to  $\mathcal{E}''$  are identical to those for  $\mathcal{E}$  along the rays generated by  $\mathbf{v}_i$  for  $1 \leq i \leq n$ . Along the new ray generated by  $\mathbf{v}_0$ , the decreasing filtration for  $\mathcal{E}''$  is  $E''^{\mathbf{v}_0}(j) = \sum_{\langle \mathbf{u}, \mathbf{v}_0 \rangle \geq j} E_{\mathbf{u}}^{\sigma_0}$ , where  $E_{\mathbf{u}}^{\sigma_0}$  is the linear subspace associated to  $\mathcal{E}$ ; see §2.2. It follows that  $\mathcal{B}(\mathcal{E}'')_{\sigma_i} = \mathcal{B}(\mathcal{E})_{\sigma_0}$  for all  $1 \leq i \leq d$ .

Finally, to analyze  $\mathcal{E}'$ , set  $\mathcal{L} := \mathcal{O}_{X'}(-D_0)$ . Writing  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d \in M$  for the minimal generators the dual cone  $\sigma^\vee$ , Example 2.8 shows that the characters associated to the line bundle  $\mathcal{L}$  are  $\mathbf{u}_{\mathcal{L}}(\sigma') = \{\mathbf{0}\}$  for all  $\sigma' \in \Sigma(d) \setminus \sigma_0$  and  $\mathbf{u}_{\mathcal{L}}(\sigma_i) = \{\mathbf{w}_i\}$  for all  $1 \leq i \leq d$ . Combining this data with that for  $\mathcal{E}''$ , we see that the characters associated to toric vector bundle  $\mathcal{E}' = \mathcal{E}'' \otimes_{\mathcal{O}_X} \mathcal{L}$  are  $\mathbf{u}_{\mathcal{E}'}(\sigma') = \mathbf{u}_{\mathcal{E}}(\sigma')$  for all  $\sigma' \in \Sigma(d) \setminus \sigma_0$  and  $\mathbf{u}_{\mathcal{E}'}(\sigma_i) = \{\mathbf{u} + \mathbf{w}_i : \mathbf{u} \in \mathbf{u}_{\mathcal{E}}(\sigma_0)\}$  for all  $1 \leq i \leq d$ . As  $\mathcal{L}$  is a line bundle, we also have  $\mathcal{B}(\mathcal{E}')_{\sigma_i} = \mathcal{B}(\mathcal{E}'')_{\sigma_i} = \mathcal{B}(\mathcal{E})_{\sigma_0}$  for all  $1 \leq i \leq d$ . If  $\mathbf{u}_{\ell} \in \mathbf{u}_{\mathcal{E}}(\sigma_0)$  corresponds to the vector  $\mathbf{e}_{\ell, \sigma_0} \in E$ , then the element  $\mathbf{u}_{\ell} + \mathbf{w}_i \in \mathbf{u}_{\mathcal{E}'}(\sigma_i)$  corresponds to the same vector in  $E$ . Since  $\mathcal{E}'$  is globally generated and  $\mathcal{E}$  is very ample, Theorem 1.2 implies that  $\mathbf{u}_{\ell} + \mathbf{w}_i \in P_{\mathbf{e}_{\ell, \sigma_0}}$  and  $\mathbf{u}_{\ell} \in P_{\mathbf{e}_{\ell, \sigma_0}}$  for all  $\mathbf{u}_{\ell} \in \mathbf{u}_{\mathcal{E}}(\sigma_0)$  and all  $1 \leq i \leq d$ . Applying Theorem 4.3, we conclude that  $\mathcal{E}$  separates 1-jets.  $\square$

*Proof of Theorem 1.3.* This follows immediately by combining Theorem 4.8 and Theorem 4.9.  $\square$

**Remark 4.10.** Combining Example 4.7 with Theorem 4.9, we see that  $\mathcal{H}$  is an ample toric vector bundle that is globally generated but not very ample, which answers the first part of Question 7.5 in [HMP]. Moreover, modifying the proof of Proposition 3.5 by replacing some non-strict inequalities with strict inequalities, we obtain a partial converse to Lemma 4.2. Specifically, if  $\mathcal{E}$  is a toric vector bundle on  $\mathbb{P}^d$  with rank at most  $d$ , then  $\mathcal{E}$  is ample if and only if it separates 1-jets. Hence,  $\mathcal{H}$  also has minimal rank among all globally-generated ample toric vector bundles on  $\mathbb{P}^2$  that are not very ample.

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